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Local dendrites with unique hyperspace $C(X)$

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ABSTRACT

For a continuum X we denote by $C(X)$ the hyperspace of subcontinua of X, metrized by the Hausdorff metric. Let $\mathfrak{D}$ be the class of dendrites whose set of end points is closed and let $\mathfrak{LD}$ be the class of local dendrites X such that every point of X has a neighborhood which is in $\mathfrak{D}$. In this paper we study the structure of the classes $\mathfrak{D}$ and $\mathfrak{LD}$. As an application, we show that if $X \in \mathfrak{LD}$ is different from an arc and a simple closed curve, and Y is a continuum such that the hyperspaces $C(X)$ and $C(Y)$ are homeomorphic, then X is homeomorphic to Y.

1. Introduction

A *continuum* is a nonempty compact, connected, metric space. All concepts not defined here will be taken as in [35]. Given a continuum X, we consider the family:

$$2^X = \{A \subset X\colon\ A \text{ is nonempty and closed}\}.$$

The topology on 2^X will be induced by the Hausdorff metric H [35, Definition 0.1, p. 1]. We also consider the following subspaces of 2^X:

$$C(X) = \left\{A \in 2^X\colon\ A \text{ is connected}\right\}, \qquad F_1(X) = \left\{\{p\}\colon\ p \in X\right\}$$

and, if $p \in X$,

$$C(p, X) = \left\{A \in C(X)\colon\ p \in A\right\}.$$

Both 2^X and $C(X)$ are called *hyperspaces of* X. It is known that 2^X and $C(X)$ are arcwise connected continua [27, Corollary 14.10, p. 114]. It is also known that $F_1(X)$ is homeomorphic to X.

A continuum X has *unique hyperspace* $C(X)$ if for every continuum Y such that the hyperspaces $C(X)$ and $C(Y)$ are homeomorphic, it follows that the continua X and Y are homeomorphic.

The topic of this paper is inserted in the following general problem.

Problem. Find conditions, on the continuum X, in order that X has unique hyperspace $C(X)$.

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A *finite graph* is a continuum that can be written as the union of finitely many arcs, each two of which are either disjoint or intersect only in one or both of their end points. A *tree* is a finite graph that contains no simple closed curves. Let

$$\mathfrak{G} = \{X\colon\ X \text{ is a finite graph}\}.$$

It has been proved that:

(a) if $X \in \mathfrak{G}$ is different from an arc and a simple closed curve, then X has unique hyperspace $C(X)$ [2, Theorem 1, p. 38].

If X is either an arc or a simple closed curve and Y is a continuum such that $C(X)$ is homeomorphic to $C(Y)$, then Y is either an arc or a simple closed curve [2, Lemma 11, p. 38].

A *dendrite* is a locally connected continuum which contains no simple closed curves. Let

$$\mathfrak{D} = \{X\colon\ X \text{ is a dendrite whose set of end points is closed}\}.$$

It has been proved that:

(b) if $X \in \mathfrak{D}$ is not an arc, then X has unique hyperspace $C(X)$ [15, Theorem 10, p. 804].

Moreover, if X is either an arc or if X is a dendrite such that $X \notin \mathfrak{D}$, then X does not have unique hyperspace $C(X)$ [7, Theorem 5.2, p. 466].

A *local dendrite* is a continuum such that every of its points has a neighborhood which is a dendrite. Let

$$\mathfrak{L} = \{X\colon\ X \text{ is a local dendrite}\}$$

and

$$\mathfrak{LD} = \{X \in \mathfrak{L}\colon\ \text{each point of } X \text{ has a neighborhood which is in } \mathfrak{D}\}.$$

One important part of this paper is dedicated to study the structure of the classes $\mathfrak{D}$ and $\mathfrak{LD}$. As an application, another important part of this paper is dedicated to prove the following result:

(c) if $X \in \mathfrak{LD}$ is different from an arc and a simple closed curve, then X has unique hyperspace $C(X)$.

Since $\mathfrak{D} \subsetneq \mathfrak{LD}$ and $\mathfrak{G} \subsetneq \mathfrak{LD}$, (c) generalizes both (a) and (b).

The paper is divided in five sections. After this section, in Section 2 we present the definitions and the fundamental results that we use in the paper. In Section 3 we study the structure of the classes $\mathfrak{D}$ and $\mathfrak{LD}$. In Theorem 3.20 we present a useful characterization of the elements of $\mathfrak{LD}$, in terms of the dimension of $C(X)$ at certain subcontinua of X. As an application of this result we prove, in Theorem 3.21, that if $X \in \mathfrak{LD}$ and $Y \in \mathfrak{L}$ are such that $C(X)$ is homeomorphic to $C(Y)$, then $Y \in \mathfrak{LD}$. This is an important first step for the proof of (c). In Section 4 we consider, for every continuum X, a class $\Omega(X)$ of special subsets of X in such a way that if X and Y are two continua so that $C(X)$ is homeomorphic to $C(Y)$, then $\Omega(X)$ is homeomorphic to $\Omega(Y)$. After presenting some additional properties of the class $\Omega(X)$, we prove in Theorem 4.10 that if $X \in \mathfrak{LD}$ is different from an arc, then the closure of $\Omega(X)$ in $C(X)$ is homeomorphic to X. This is an important second step for the proof of (c). Finally in Section 5 we prove (c).

Results related to the subject of this paper can be found in [1–7,9,14–18,20–26,30–33].

2. Definitions and fundamental results

The symbol $\mathbb{N}$ will denote the set of positive integers. Let Z be a metric space and A be a subset of Z. We denote by $|A|$ the cardinality of A, and by $\operatorname{diam}(A)$ the diameter of A. The interior, the closure and the boundary of A in Z, will be denoted by $\operatorname{Int}_Z(A)$, $\operatorname{Cl}_Z(A)$ and $\operatorname{Bd}_Z(A)$, respectively. If $\varepsilon > 0$, we will use the set $N(\varepsilon, A) = \bigcup_{p\in A} B_Z(p,\varepsilon)$ where, for $p \in Z$, $B_Z(p,\varepsilon)$ denotes the open ε-ball in Z centered at p. If $p \in Z$, then $\dim_p(Z)$ stands for the dimension of Z at p [37, p. 5]. The following two results are proved in [15, Lemma 5, p. 803] and [28, Theorem 4.4, p. 28], respectively.

Theorem 2.1. *Let X be a locally connected continuum and $A \in C(X)$. If $\dim_A(C(X)) < \infty$, then $\dim_{\{p\}}(C(X)) < \infty$, for every $p \in A$.*

Theorem 2.2. *Let X be a continuum. Then X is locally connected if and only if $C(X)$ is locally connected.*

For $n \in \mathbb{N}$, an *n-cell* is a space homeomorphic to the Cartesian product $\prod_{i=1}^{n} X_i$ where $X_i = [0,1]$, for each $i \in \{1, 2, \dots, n\}$. A 1-cell is called an *arc*. A *Hilbert cube* is a space homeomorphic to the Cartesian product $\prod_{i=1}^{\infty} X_i$ where $X_i = [0,1]$, for each $i \in \mathbb{N}$. The following result is proved in [13, Theorem 4, p. 221].

Theorem 2.3. *Let X be a continuum and $p \in X$. If X is locally connected at each point of an open set containing p, then $C(p, X)$ is a Hilbert cube if and only if p is not in the interior* (*relative to X*) *of a finite graph in X.*

For $n \in \mathbb{N} - \{1, 2\}$ an *n-od* is a continuum Y which contains a subcontinuum Z such that $Y - Z = \bigcup_{i=1}^{n} Z_i$, where $Z_i \neq \emptyset$ for each $i \in \{1, 2, \ldots, n\}$ and $\mathrm{Cl}_Y(Z_i) \cap Z_j = \emptyset$ whenever $i \neq j$. The following result is proved in [2, Lemma 8, p. 37].

Theorem 2.4. *Let X be a continuum and $n \in \mathbb{N} - \{1, 2\}$. If $K \in C(X)$ and T is an n-od in X such that, for some $\varepsilon > 0$, $T \in B_{C(X)}(K, \frac{\varepsilon}{2})$, then there is an n-cell Γ in $C(X)$ such that $T \in \Gamma \subset B_{C(X)}(K, \varepsilon)$.*

If V is a 2-cell in a space X, then ∂V denotes the *manifold boundary* of V. Note that if V is a 2-cell and $h : [0, 1]^2 \to V$ is a homeomorphism, then $\partial V = h(\mathrm{Bd}_{\mathbb{R}^2}([0, 1]^2))$ [11, Theorem 17.A.8, p. 449]. If A and B are 2-cells such that $A \subset B$, then $A - \partial A$ is open in B [37, 19.34, p. 123]. If A is an arc with end points p and q then, by [27, Example 5.1.1, p. 33], $C(A)$ is a 2-cell such that $\partial C(A) = C(p, A) \cup C(q, A) \cup F_1(A)$.

Let Z be a dendrite. It is known that every subcontinuum of Z is a dendrite [36, Corollary 10.6, p. 167]. It is also known that every two points x and y in Z, can be joined by a unique arc contained in Z. We denote such arc by $[x, y]$, and consider that $[x, x] = \{x\}$. We define $(x, y) = [x, y] - \{x, y\}$, $[x, y) = [x, y] - \{y\}$ and $(x, y] = [x, y] - \{x\}$. If Z is a local dendrite and $x, y \in Z$, then the symbol $[x, y]$ will represent an arc in Z with end points x and y. The sets (x, y), $[x, y)$ and $(x, y]$ are defined as before.

A metric space Y, with metric d, is *uniformly locally arcwise connected* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in X$ with $x \neq y$ and $d(x, y) < \delta$, there is an arc A in Y with end points x and y, so that $\mathrm{diam}(A) < \varepsilon$. It is known that locally connected continua are uniformly locally arcwise connected [19, Lemma 3-29, p. 129]. In particular dendrites are uniformly locally arcwise connected.

The following result, which we will use in Section 3, was presented in [15, Lemma 3, p. 802] without proof. For the readers convenience, we prove it in this paper.

Theorem 2.5. *Let X be a dendrite, $p \in X$, $M \in C(X)$ and $[a, b]$ be a nondegenerate arc in X such that $p \in (a, b) \subset [a, b] \subset M$. Suppose that a sequence $\{M_n\}_{n\in\mathbb{N}}$ in $C(X)$ converges, in the Hausdorff metric, to M. Then there is an $N \in \mathbb{N}$ such that $p \in M_n$, for every $n > N$.*

Proof. Let $\varepsilon > 0$ be such that the sets $B_X(a, \varepsilon)$, $B_X(b, \varepsilon)$ and $B_X(p, \varepsilon)$ are mutually disjoint. Let d be the metric of X. Since X is uniformly locally arcwise connected, there exists $\delta > 0$ such that for every $x, y \in X$ with $x \neq y$ and $d(x, y) < \delta$, there is an arc A in X with end points x and y, so that $\mathrm{diam}(A) < \varepsilon$. Since $\{M_n\}_{n\in\mathbb{N}}$ converges to M, there is $N \in \mathbb{N}$ such that $H(M_n, M) < \delta$, for every $n \geqslant N$. Now assume that $n > N$. Since $a, b \in [a, b] \subset M \subset N(\delta, M_n)$, there exist $a_1 \in M_n \cap B_X(a, \delta)$ and $b_1 \in M_n \cap B_X(b, \delta)$. Then $d(a, a_1) < \delta$ and $d(b, b_1) < \delta$, so there exist an arc A with end points a and a_1, and an arc B with end points b and b_1, so that $\mathrm{diam}(A) < \varepsilon$ and $\mathrm{diam}(B) < \varepsilon$. Note that $A \subset B_X(a, \varepsilon)$ and $B \subset B_X(b, \varepsilon)$. Let $a_2 \in A$ be so that $[a_1, a_2] \cap [a, b] = \{a_2\}$. We define $b_2 \in B$ so that $[b_1, b_2] \cap [a, b] = \{b_2\}$. Note that $C = [a_1, a_2] \cup [a_2, b_2] \cup [b_2, b_1]$ is an arc in X that contains p, and whose end points a_1 and b_1 are in M_n. Thus, $C \subset M_n$ and then $p \in M_n$. □

Let Y be a continuum, $p \in Y$ and β be a cardinal number. We say that p has *order less than or equal to β in Y*, written $\mathrm{ord}(p, Y) \leqslant \beta$, provided that p has a basis $\mathfrak{B}$ of neighborhoods in Y such that, for each $U \in \mathfrak{B}$, we have $|\mathrm{Bd}_Y(U)| \leqslant \beta$. We say that p has *order equal to β in Y*, written $\mathrm{ord}(p, Y) = \beta$, if $\mathrm{ord}(p, Y) \leqslant \beta$ and $\mathrm{ord}(p, Y) \nleqslant \alpha$, for each cardinal number α such that $\alpha < \beta$.

Given a continuum Y, we consider the following sets:

$$E(Y) = \{p \in Y\colon \mathrm{ord}(p, Y) = 1\}, \qquad O(Y) = \{p \in Y\colon \mathrm{ord}(p, Y) = 2\}$$

and

$$R(Y) = \{p \in Y\colon \mathrm{ord}(p, Y) \geqslant 3\}.$$

The elements of $E(Y)$, $O(Y)$ and $R(Y)$ are called *end points, ordinary points* and *ramification points* of Y, respectively. We also consider the following subset of Y:

$$E_a(Y) = \{p \in Y\colon \text{there exists a convergent sequence } \{e_n\}_{n\in\mathbb{N}} \text{ in } E(Y) - \{p\} \text{ whose limit is } p\}.$$

Let Z be a dendrite. Then $p \in E(Z)$ if and only if p is an end point of every arc in Z that contains p (see [29, Theorem 15, p. 320] and [36, 10.44, p. 188]). From this and the fact that every subcontinuum of Z is a dendrite, it follows that if $A \in C(Z)$, then $E(Z) \cap A \subset E(A)$.

Let Z be a dendrite. If $p \in Z$ and $\mathrm{ord}(p, Z)$ is finite, then it is equal to the number of components of $Z - \{p\}$ [38, (1.1)(iv), p. 88]. If $\mathrm{ord}(p, Z)$ is infinite, then it is countable and the diameters of components of $Z - \{p\}$ tend to zero [38, (2.6), p. 92]. In this case we say that p is a *I-essential point of Z*. We say that p is a *II-essential point of Z*, if there exist a nondegenerate arc A in Z and a convergent sequence $\{p_n\}_{n\in\mathbb{N}}$ of different points in A, whose limit is p, and such that $p_n \in R(Z)$ for each $n \in \mathbb{N}$.

Let X be a local dendrite and $p \in X$. We say that p is a *I-essential point of* X (respectively, a *II-essential point of* X) if there exists a dendrite Z which is a neighborhood of p in X, such that p is a *I*-essential point of Z (respectively, a *II*-essential point of Z). We say that p is an *essential point of* X if p is either a *I*-essential or a *II*-essential point of X.

The following result, easy to prove, is the equivalent version of [34, Properties 1.2(e), p. 43] for local dendrites.

Theorem 2.6. *Let X be a local dendrite and $p \in X$. Then p is an essential point of X if and only if $p \notin \mathrm{Int}_X(T)$, for every tree T contained in X.*

Theorem 2.7. *Let X be a local dendrite and $A \in C(X)$. If there exists $p \in A$ such that p is an essential point of X, then $\dim_A(C(X)) = \infty$.*

Proof. By Theorem 2.6, p is not in the interior, relative to X, of a finite graph in X. Then, by Theorem 2.3, $C(p, X)$ is a Hilbert cube. Hence, $\dim_A(C(p, X)) = \infty$. Since $A \in C(p, X) \subset C(X)$, it follows from [37, Corollary 3.3, p. 16], that $\dim_A(C(p, X)) \leqslant \dim_A(C(X))$. Thus, $\dim_A(C(X)) = \infty$. $\square$

If $a, b \in \mathbb{R}^2$, then $\overline{ab}$ denotes the straight line segment joining a and b. We will need three special dendrites F_ω, W and W_0, constructed in $\mathbb{R}^2$. The first one is the dendrite

$$F_\omega = \bigcup_{n\in\mathbb{N}} \overline{pp_n}, \tag{2.1}$$

where $p = (0, 0)$ and $p_n = (\frac{1}{n}, \frac{1}{n^2})$, for each $n \in \mathbb{N}$. We say that p is the *vertex* of F_ω. The second one is the dendrite

$$W = \overline{cb_1} \cup \left(\bigcup_{n\in\mathbb{N}} \overline{a_n b_n} \right), \tag{2.2}$$

where $c = (-1, 0)$, $a_n = (\frac{1}{n}, \frac{1}{n})$ and $b_n = (\frac{1}{n}, 0)$, for each $n \in \mathbb{N}$. The third one is the dendrite

$$W_0 = \overline{bb_1} \cup \left(\bigcup_{n\in\mathbb{N}} \overline{a_n b_n} \right), \tag{2.3}$$

where $b = (0, 0)$ and, for each $n \in \mathbb{N}$, a_n and b_n are defined as before. Note that $W_0 = W - [c, b)$.

3. The class $\mathfrak{LD}$

We recall that $\mathfrak{D}$ is the class of dendrites whose set of end points is closed. Let $X \in \mathfrak{D}$. By [8, Theorem 3.3, p. 4], the order of every point of X is finite. The following result follows from the proof of [8, Proposition 3.4, p. 4].

Theorem 3.1. *Let $X \in \mathfrak{D}$ and $p \in X$. If p is the limit of a sequence $\{p_n\}_{n\in\mathbb{N}}$ of distinct ramification points of X such that $p \neq p_1$, then p is both the limit of a sequence of distinct ramification points of X, all in the arc $[p, p_1]$, and the limit of a sequence of end points of X, all different from p.*

If X is a dendrite then, by [10, Corollary 4, p. 298], X satisfies the following property:

(S) for every $e \in X$ and every sequence $\{p_n\}_{n\in\mathbb{N}}$ in X that converges to p, the sequence of arcs $\{[e, p_n]\}_{n\in\mathbb{N}}$ converges, in the Hausdorff metric, to the arc $[e, p]$.

Theorem 3.2. *Let $X \in \mathfrak{D}$ and $e \in X$. If e is the limit of a sequence $\{e_n\}_{n\in\mathbb{N}}$ of distinct end points of X such that $e \neq e_1$, then there exist an increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ in $\mathbb{N}$ and a sequence $\{p_m\}_{m\in\mathbb{N}}$ of distinct ramification points of X, all in the arc $[e, e_1]$, such that e is the limit of $\{p_m\}_{m\in\mathbb{N}}$ and*

$$[e, p_1] \cup \left(\bigcup_{m\in\mathbb{N}} [e_{n_m}, p_m] \right)$$

is a subcontinuum of X, homeomorphic to the dendrite W_0 defined in (2.3).

Proof. Since $X \in \mathfrak{D}$, the set $E(X)$ is closed in X. Then $e \in E(X)$. We can assume, without loss of generality, that $e \neq e_n$ for every $n \in \mathbb{N}$. Recall that $p \in E(X)$ if and only if p is an end point of every arc in X that contains p. Hence, $e_n \notin [e, e_1]$, for every $n \in \mathbb{N} - \{1\}$. Let $p_1 \in [e, e_1]$ be so that $[e_2, p_1] \cap [e, e_1] = \{p_1\}$. Note that $p_1 \in R(X)$, so $[e, p_1] \neq \{e\}$. Let $n_1 = 2$. If for infinitely many indices n we have $[e, p_1] \cap [e, e_n] = [e, p_1]$, then for such indices n it follows that $[e, p_1] \subset [e, e_n]$. Taking the limit we obtain, by the property (S), that $[e, p_1] \subset [e, e] = \{e\}$, so $[e, p_1] = \{e\}$. Since this is a contradiction we have shown that:

1) there exists $n_2 > n_1$ such that $[e, p_1] \cap [e, e_n] \neq [e, p_1]$, for each $n \geqslant n_2$.

Now we claim that:

2) $[e, e_n] \cap [e_2, p_1] = \emptyset$, for each $n \geqslant n_2$.

To show 2) assume, on the contrary, that $[e, e_n] \cap [e_2, p_1] \neq \emptyset$, for some $n \geqslant n_2$. By compactness of the set $[e, e_n] \cap [e_2, p_1]$, there exists $b \in [e_2, p_1]$ such that $[e_n, b] \cap [e_2, p_1] = \{b\}$. Since any two points of X can be joined by a unique arc in X, the arc $[e, e_n]$ coincides with the arc $[e, p_1] \cup [p_1, b] \cup [e_n, b]$. Then $b \in [e, e_n]$ and $[e, p_1] \cap [e, e_n] = [e, p_1]$. Since this contradicts 1), claim 2) is true.

Since X is a dendrite for every $n \geqslant n_2$, by 1) and [36, Theorem 10.10, p. 169], $[e, p_1] \cap [e, e_n]$ is a proper subarc of $[e, p_1]$. Let $p_2 \in [e, p_1]$ be so that $[e_{n_2}, p_2] \cap [e, p_1] = \{p_2\}$. Then $p_2 \in R(X) - \{p_1\}$, $[e, e_{n_2}] = [e, p_2] \cup [e_{n_2}, p_2]$ and, since $[e, e_{n_2}] \cap [e_2, p_1] = \emptyset$, we have $[e_{n_2}, p_2] \cap [e_{n_1}, p_1] = \emptyset$.

Following the argument that lead us to 1), we can show that there exists $n_3 > n_2$ such that $[e, p_2] \cap [e, e_n] \neq [e, p_2]$, for each $n \geqslant n_3$. Using the arguments that lead us to 2), it follows that $[e, e_n] \cap [e_{n_1}, p_1] = \emptyset$ and $[e, e_n] \cap [e_{n_2}, p_2] = \emptyset$, for each $n \geqslant n_3$. Let $p_3 \in [e, p_2]$ be so that $[e_{n_3}, p_3] \cap [e, p_2] = \{p_3\}$. Then $p_3 \in R(X) - \{p_1, p_2\}$, $[e, e_{n_3}] = [e, p_3] \cup [e_{n_3}, p_3]$, $[e_{n_3}, p_3] \cap [e_{n_1}, p_1] = \emptyset$ and $[e_{n_3}, p_3] \cap [e_{n_2}, p_2] = \emptyset$.

Proceeding in this fashion we can obtain an increasing sequence $\{n_m\}_{m \in \mathbb{N}}$ of natural numbers and a sequence $\{p_m\}_{m \in \mathbb{N}}$ in $R(X)$ with the following properties:

(a) $p_1 \in [e, e_1)$ and $[e_{n_1}, p_1] \cap [e, e_1] = \{p_1\}$;
(b) for $m \geqslant 2$, we have $p_m \in [e, p_{m-1})$, $[e_{n_m}, p_m] \cap [e, p_{m-1}] = \{p_m\}$ and $[e_{n_m}, p_m] \cap [e_{n_i}, p_i] = \emptyset$, for every $i \in \{1, 2, \ldots, m-1\}$.

Hence, $\{p_m\}_{m \in \mathbb{N}}$ is a sequence of distinct ramification points of X, all in the arc $[e, e_1]$. Given $m \in \mathbb{N}$ we have $p_m \in [e_{n_m}, p_m] \subset [e, e_{n_m}]$. Since the sequence $\{e_{n_m}\}_{m \in \mathbb{N}}$ converges to e, by the property (S), the sequence of arcs $\{[e, e_{n_m}]\}_{m \in \mathbb{N}}$ converges, in the Hausdorff metric, to $\{e\}$. This implies that the sequence of pairwise disjoint arcs $\{[e_{n_m}, p_m]\}_{m \in \mathbb{N}}$ also converges, in the Hausdorff metric, to $\{e\}$. Then, e is the limit of $\{p_m\}_{m \in \mathbb{N}}$ and

$$[e, p_1] \cup \left(\bigcup_{m \in \mathbb{N}} [e_{n_m}, p_m] \right)$$

is a subcontinuum of X, homeomorphic to the dendrite W_0. $\square$

For a continuum Y we recall that $e \in E_a(Y)$ if and only if there exists a convergent sequence $\{e_n\}_{n \in \mathbb{N}}$ in $E(Y) - \{e\}$, whose limit is e. Combining Theorems 3.1 and 3.2 we obtain the following result.

Theorem 3.3. *Let $X \in \mathfrak{D}$ and $e \in X$. Then $e \in E_a(X)$ if and only if e is the limit of a sequence of distinct ramification points of X.*

Now recall that $\mathfrak{L}$ is the class of local dendrites, and that $\mathfrak{LD}$ is the class of local dendrites with the property that each of its points has a neighborhood in $\mathfrak{D}$. For $X \in \mathfrak{LD}$ in this section we present two characterizations of the essential points of X. We also present a characterization of the elements of $\mathfrak{LD}$, in terms of the dimension of $C(X)$ at certain subcontinua of X (see Theorem 3.20).

Theorem 3.4. *Let $X \in \mathfrak{L}$. Then $X \in \mathfrak{LD}$ if and only if X contains no copy of F_ω or of W (defined in* (2.1) *and* (2.2)*).*

Proof. In [8, Theorem 3.3, p. 4] it is proved that a dendrite is in $\mathfrak{D}$ if and only if it contains no copy of F_ω or of W. The result follows from this fact. $\square$

The next result is proved in [29, Theorem 4, p. 303].

Theorem 3.5. *The following conditions are equivalent*:

(a) *X is a local dendrite*;
(b) *X is a locally connected continuum which contains at most a finite number of simple closed curves*;
(c) *X is a continuum and there exists a finite number of dendrites $D_1, D_2, \ldots, D_l$ such that $X = D_1 \cup D_2 \cup \cdots \cup D_l$ and $|D_i \cap D_j| < \infty$ for every $i, j \in \{1, 2, \ldots, l\}$ with $i \neq j$.*

Corollary 3.6. *If $X \in \mathfrak{L}$ and $Z \in C(X)$, then $Z \in \mathfrak{L}$. If $X \in \mathfrak{LD}$ and $Z \in C(X)$, then $Z \in \mathfrak{LD}$.*

Proof. Assume first that $X \in \mathfrak{L}$ and $Z \in C(X)$. Then, by [29, Theorem 1, p. 303] and [29, Theorem 2, p. 283], X is hereditarily locally connected. Thus, Z is a locally connected continuum and, by Theorem 3.5, Z is a local dendrite.

Assume now that $X \in \mathfrak{LD}$ and $Z \in C(X)$. By the first part of the proof, $Z \in \mathfrak{L}$ and, by Theorem 3.4, $Z \in \mathfrak{LD}$. □

The next result follows from Theorems 3.4 and 3.5.

Theorem 3.7. *Let X be a continuum that contains no copy of F_ω or of W. Then $X \in \mathfrak{LD}$ if and only if there exists a finite number of dendrites $D_1, D_2, \ldots, D_l$ in $\mathfrak{D}$ such that $X = D_1 \cup D_2 \cup \cdots \cup D_l$ and $|D_i \cap D_j| < \infty$ for every $i, j \in \{1, 2, \ldots, l\}$ with $i \neq j$.*

From now on if $X \in \mathfrak{L}$ (respectively, if $X \in \mathfrak{LD}$) we will think that X is nondegenerate and that

$$X = D_1 \cup D_2 \cup \cdots \cup D_l, \tag{3.1}$$

where $D_1, D_2, \ldots, D_l$ are nondegenerate dendrites (respectively, $D_1, D_2, \ldots, D_l$ are nondegenerate elements of $\mathfrak{D}$) such that $|D_i \cap D_j| < \infty$ for every $i, j \in \{1, 2, \ldots, l\}$ with $i \neq j$. We will also consider that:

$$P_X = \bigcup \bigl\{D_i \cap D_j \colon\ i, j \in \{1, 2, \ldots, l\} \text{ and } i \neq j\bigr\}.$$

Note that P_X is a finite subset of X. Note also that if $p \in P_X$, then $p \notin E(X)$ and if $p \in D_i - P_X$, then $\operatorname{ord}(p, D_i) = \operatorname{ord}(p, X)$.

Theorem 3.8. *If $X \in \mathfrak{L}$ and $i \in \{1, 2, \ldots, l\}$, then $D_i - P_X$ is open in X.*

Proof. Let $p \in D_i - P_X$. Since X is locally connected and P_X is finite, there exists an open and connected subset U of X such that $p \in U \subset X - P_X$. Then U is a connected subset of X such that $C \cap D_i \neq \emptyset$ and $C \cap P_X = \emptyset$, so $U \subset D_i - P_X$. This shows that $D_i - P_X$ is open in X. □

Theorem 3.9. *If $X \in \mathfrak{L}$ and $p \in D_i \cap P_X$, then there exists a nondegenerate arc A in D_i such that $A \cap P_X = \{p\}$.*

Proof. Since P_X is finite there exists $\varepsilon > 0$ such that $B_X(p, \varepsilon) \cap P_X = \{p\}$. Since D_i is uniformly locally arcwise connected, if d denotes the metric on X, there is $\delta > 0$ such that for every $x, y \in D_i$ with $x \neq y$ and $d(x, y) < \delta$, there is an arc A in D_i with end points x and y, so that $\operatorname{diam}(A) < \varepsilon$. Taking $x \in D_i$ such that $x \neq p$ and $d(x, p) < \delta$, it then follows that there exists an arc A in D_i with end points x and p, so that $\operatorname{diam}(A) < \varepsilon$. Then, $A \subset B_X(p, \varepsilon)$, so $A \cap P_X = \{p\}$. □

Theorem 3.10. *If $X \in \mathfrak{LD}$, then $E_a(D_i) \cap P_X = \emptyset$, for every $i \in \{1, 2, \ldots, l\}$.*

Proof. Assume, on the contrary, that there exists $e \in E_a(D_i) \cap P_X$. Then e is the limit of a sequence $\{e_n\}_{n\in\mathbb{N}}$ in $E(D_i) - \{e\}$. If there exists an infinite subset J of $\mathbb{N}$ such that $e_n = e_m$, for every $n, m \in J$, then $e = e_n$, for some $n \in \mathbb{N}$. Since this is a contradiction, we can assume that $e_n \neq e_m$ if $n \neq m$. By Theorem 3.2, there exist an increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ in $\mathbb{N}$ and a sequence $\{p_m\}_{m\in\mathbb{N}}$ of distinct ramification points of D_i, all in the arc $[e, e_1]$ of D_i, such that e is the limit of $\{p_m\}_{m\in\mathbb{N}}$ and

$$F = [e, p_1] \cup \left(\bigcup_{m\in\mathbb{N}} [e_{n_m}, p_m] \right)$$

is a subcontinuum of D_i, homeomorphic to the dendrite W_0 defined in (2.3). Since $e \in P_X$, there exists $k \in \{1, 2, \ldots, l\} - \{i\}$ such that $e \in D_i \cap D_k$. Then $e \in D_k \cap P_X$ so, by Theorem 3.9, there exists a nondegenerate arc A is D_k such that $A \cap P_X = \{e\}$. Then $A \cup F$ is a copy of W. This implies that X contains a copy of W. Since this contradicts Theorem 3.4, it follows that $E_a(D_i) \cap P_X = \emptyset$. □

As a consequence of the previous theorem, we have the following result.

Theorem 3.11. *If $X \in \mathfrak{LD}$, then $E(X)$ is closed in X. In particular, $E_a(X) \subset E(X)$.*

Proof. To show that $E(X)$ is closed in X, let e be the limit of a sequence $\{e_n\}_{n\in\mathbb{N}}$ in $E(X)$. We can assume that $e_n \neq e$ for every $n \in \mathbb{N}$. If there exists an infinite subset J of $\mathbb{N}$ such that $e_n = e_m$, for every $n, m \in J$, then $e \in E(X)$. Hence, without loss of generality, we can assume that $e_n \neq e_m$ if $n \neq m$. By (3.1), we can also assume that there exists $i \in \{1, 2, \ldots, l\}$ such that $e_n \in D_i$ for every $n \in \mathbb{N}$. Hence, $e \in D_i$. Since $P_X \subset X - E(X)$ we have $e_n \in D_i - P_X$ so $\operatorname{ord}(e_n, D_i) = \operatorname{ord}(e_n, X) = 1$, for all $n \in \mathbb{N}$. Since $D_i \in \mathfrak{D}$, the set $E(D_i)$ is closed in D_i. Hence, $e \in E(D_i)$ and, indeed, $e \in E_a(D_i)$ so, by Theorem 3.10, $e \notin P_X$. Hence, $\operatorname{ord}(e, X) = \operatorname{ord}(e, D_i) = 1$. This shows that $E(X)$ is closed in X. In particular, $E_a(X) \subset E(X)$. □

The following result is the equivalent version of Theorem 3.3, for the elements of $\mathfrak{LD}$.

Theorem 3.12. *Let $X \in \mathfrak{LD}$ and $e \in X$. Then the following assertions are equivalent*:

(a) $e \in E_a(X)$;
(b) *e is the limit of a sequence of distinct ramification points of X, all in one arc of X that contains e*;
(c) *e is the limit of a sequence of distinct ramification points of X.*

Proof. To show that (a) implies (b), let us assume that $e \in E_a(X)$. By (3.1), there is $i \in \{1, 2, \ldots, l\}$ such that $e \in D_i$. Since $e \in E(X)$ and $P_X \subset X - E(X)$, we have $e \in D_i - P_X$. Then $\operatorname{ord}(e, D_i) = \operatorname{ord}(e, X) = 1$. Let $\{e_n\}_{n\in\mathbb{N}}$ be a sequence in $E(X) - \{e\}$ whose limit is e. We can consider that $e_n \neq e_m$ if $n \neq m$. Since $D_i - P_X$ is open in X (Theorem 3.8), we can also consider that $e_n \in D_i - P_X$, for every $n \in \mathbb{N}$, so $\operatorname{ord}(e_n, D_i) = \operatorname{ord}(e_n, X) = 1$. Hence, e is the limit of a sequence of distinct end points of D_i and, by Theorem 3.2, e is also the limit of a sequence of distinct ramification points of D_i, all in one arc of D_i that contains e. Since every ramification point of D_i is a ramification point of X, this shows that (a) implies (b).

The assertion (b) implies (c) is obvious. To show that (c) implies (a), let us assume that e is the limit of a sequence $\{p_n\}_{n\in\mathbb{N}}$ of distinct ramification points of X. Then $p_n \neq p_m$ if $n \neq m$ and we can consider that $p_n \neq e$ for every $n \in \mathbb{N}$. Let us assume first that $e \notin P_X$. By (3.1), there is $i \in \{1, 2, \ldots, l\}$ such that $e \in D_i$. Since $D_i - P_X$ is open in X (Theorem 3.8), we can consider that $p_n \in D_i - P_X$, for every $n \in \mathbb{N}$. Thus, $\operatorname{ord}(p_n, D_i) = \operatorname{ord}(p_n, X) \geqslant 3$ so each p_n is a ramification point of D_i. By Theorem 3.1, e is the limit of a sequence $\{e_n\}_{n\in\mathbb{N}}$ of end points of D_i, all different from e. We can then assume that $e_n \neq e_m$ if $n \neq m$. Since P_X is finite, we can also assume that $e_n \notin P_X$, for every $n \in \mathbb{N}$. Thus, $\operatorname{ord}(e_n, X) = \operatorname{ord}(e_n, D_i) = 1$, for each $n \in \mathbb{N}$ and then $e \in E_a(X)$.

Let us consider now that $e \in P_X$. By (3.1), we can assume that there is $i \in \{1, 2, \ldots, l\}$ such that $p_n \in D_i$ for each $n \in \mathbb{N}$. Then $e \in D_i$. If $p_n \in P_X$ for infinitely many indices n then, since P_X is finite and $p_n \neq e$ for every $n \in \mathbb{N}$, there exists $q \in P_X - \{e\}$ such that $q = p_n$ for infinitely many indices n. Thus, $e = q$ and, since this is a contradiction, we can assume that $p_n \notin P_X$ for every $n \in \mathbb{N}$. Hence, $\operatorname{ord}(p_n, D_i) = \operatorname{ord}(p_n, X) \geqslant 3$, for each $n \in \mathbb{N}$. By Theorem 3.1, $e \in E_a(D_i)$. Hence, $E_a(D_i) \cap P_X \neq \emptyset$. Since this contradicts Theorem 3.10, we conclude that the case $e \in P_X$ is not possible. This shows that (c) implies (a). □

Corollary 3.13. *If $X \in \mathfrak{LD}$, then $O(X)$ is open in X.*

Proof. Let $p \in O(X)$ and assume, on the contrary, that $O(X)$ is not open. Then, for each $n \in \mathbb{N}$, there exists $p_n \in B_X(p, \frac{1}{n})$ such that $p_n \in X - O(X) = E(X) \cup R(X)$. Note that p is the limit of the sequence $\{p_n\}_{n\in\mathbb{N}}$. If there exists an infinite subset J of $\mathbb{N}$ such that $p_n = p_m$ for every $n, m \in J$, then we have $p \in E(X) \cup R(X)$. Since this contradicts the fact that $p \in O(X)$, we can consider that $p_n \neq p_m$ if $n \neq m$. If for infinitely many indices n, we have $p_n \in E(X)$ then, since $E(X)$ is closed in X, we have $p \in E(X)$. Since this contradicts the fact that $p \in O(X)$, it follows that $p_n \in E(X)$ only for finitely many indices n. Hence, we can assume that $p_n \in R(X)$ for every $n \in \mathbb{N}$. Then, by Theorems 3.11 and 3.12, $p \in E_a(X) \subset E(X)$. Since this contradicts the fact that $p \in O(X)$, we conclude that $O(X)$ is open in X. □

Corollary 3.14. *Let $X \in \mathfrak{LD}$ and $A \in C(X)$. Then*:

(a) $E_a(A) \subset E_a(X)$;
(b) *if $A \cap E_a(X) = \emptyset$, then A is a finite graph.*

Proof. By Corollary 3.6, $A \in \mathfrak{LD}$. To prove (a) let $e \in E_a(A)$. Then, by Theorem 3.12, e is the limit of a sequence of distinct ramification points of A. Since a ramification point of A is also a ramification point of X, e is the limit of a sequence of distinct ramification points of X. Then, by Theorem 3.12, $e \in E_a(X)$. This shows (a).

To show (b) let us assume that $A \cap E_a(X) = \emptyset$. By Theorem 3.4, X contains no copy of F_ω. Hence, the order of every point of X is finite. In particular the order of every point of A is finite. If we assume that A is not a finite graph then, by [36, Theorem 9.10, p. 144], A contains infinitely many ramification points. Since A is compact, there is $e \in A$ such that e is the limit of a sequence of distinct ramification points of A. Then, by Theorem 3.12 and (a), $e \in E_a(A) \subset E_a(X)$ so $A \cap E_a(X) \neq \emptyset$. This contradiction shows that A is a finite graph. □

Corollary 3.15. *If $X \in \mathfrak{LD}$, then*

$$E_a(X) = \{p \in X\colon \ p \text{ is a } II\text{-essential point of } X\}.$$

Proof. Suppose that $p \in X$ is a *II*-essential point of X. Then there exists a dendrite $V \in \mathfrak{D}$, which is a neighborhood of p in X, such that p is a *II*-essential point of V. By [15, Proposition 3(a), p. 799], $p \in E_a(V)$ and by part (a) of Corollary 3.14, $p \in E_a(X)$.

Suppose now that $p \in E_a(X)$. Then, by Theorem 3.12, p is the limit of a sequence of distinct ramification points of X, all in one arc of X. From this it follows that p is a *II*-essential point of X. □

From Theorem 3.4 if $X \in \mathfrak{LD}$, then the order of every point in X is finite. Thus, X contains no I-essential points. This implies that every essential point of X is a II-essential point of X and, by Corollary 3.15, is also an element of $E_a(X)$. Hence, $E_a(X)$ is precisely the set of essential points of X.

Theorem 3.16. *Let $X \in \mathfrak{LD}$ and $A \in C(X)$. Then there exists $p \in A$ such that p is an essential point of X if and only if $\dim_A(C(X)) = \infty$.*

Proof. Suppose $p \in A$ is an essential point of X. Since $X \in \mathfrak{L}$, by Theorem 2.7, $\dim_A(C(X)) = \infty$. Now suppose that A does not contain essential points of X. Then, by Corollary 3.15, $A \cap E_a(X) = \emptyset$. Hence, by part (b) of Corollary 3.14, A is a finite graph. Let G be a finite graph in X such that $A \subset \mathrm{Int}_X(G)$ and $G \cap E_a(X) = \emptyset$. By [12, 7.4, p. 278], $\dim(C(G)) < \infty$. Thus, if $m = \dim_A(C(G))$, then $m < \infty$. Now we show that $m = \dim_A(C(X))$, so let $\mathcal{W}$ be an open subset of $C(X)$ such that $A \in \mathcal{W}$. Since $A \subset \mathrm{Int}_X(G)$, there exists $\varepsilon > 0$ such that $N(\varepsilon, A) \subset \mathrm{Int}_X(G)$. Thus, $B_{C(X)}(A, \varepsilon) \subset C(G)$, so $A \in \mathrm{Int}_{C(X)}(C(G))$. Let $\mathcal{V}$ be an open subset of $C(G)$ such that $A \in \mathcal{V} \subset \mathcal{W} \cap \mathrm{Int}_{C(X)}(C(G))$ and $|\mathrm{Bd}_{C(G)}(\mathcal{V})| = m$. Since $|\mathrm{Bd}_{C(X)}(\mathcal{V})| = |\mathrm{Bd}_{C(G)}(\mathcal{V})|$, it follows that $\dim_A(C(X)) = m$, so $\dim_A(C(X)) < \infty$. □

If D is a dendrite then, by [29, Theorem 8, p. 302], $O(D)$ is dense in D and, by [36, Theorem 10.23, p. 174], $R(D)$ is countable.

Theorem 3.17. *If $X \in \mathfrak{L}$, then $R(X)$ is countable and $O(X)$ is dense in X.*

Proof. Recall that $X = D_1 \cup D_2 \cup \cdots \cup D_l$, where $D_1, D_2, \ldots, D_l$ are nondegenerate dendrites such that $|D_i \cap D_j| < \infty$ for every $i, j \in \{1, 2, \ldots, l\}$ with $i \neq j$. It is not difficult to prove that

$$R(X) = \big(R(X) \cap P_X\big) \cup \left(\bigcup_{i=1}^{l} \big(R(D_i) - P_X\big)\right).$$

Since each set $R(D_i)$ is countable and P_X is finite, by the above equality, $R(X)$ is countable. To show that $O(X)$ is dense in X, let $p \in X$ and U be an open set in X such that $p \in U$. Let $i \in \{1, 2, \ldots, l\}$ be such that $p \in D_i$. Assume that $p \notin P_X$. Since P_X is finite and X is locally connected, there exists an open and connected subset V of X such that $p \in V \subset U \cap (X - P_X)$. Then $V \subset D_i$ and, since $O(D_i)$ is dense in D_i, there is $q \in V \cap O(D_i)$. Since $V \cap P_X = \emptyset$, we have $\mathrm{ord}(q, X) = \mathrm{ord}(q, D_i) = 2$, so $U \cap O(X) \neq \emptyset$. Now assume that $p \in P_X$. Taking a sequence of distinct ordinary points of D_i that converges to p, we infer that there exists $q \in U \cap O(D_i)$ such that $q \notin P_X$. Then $\mathrm{ord}(q, X) = \mathrm{ord}(q, D_i) = 2$, so $U \cap O(X) \neq \emptyset$. □

The following result generalizes Theorem 2.5.

Theorem 3.18. *Let $X \in \mathfrak{L}$, $p \in X$, $M \in C(X)$ and $[a, b]$ be a nondegenerate arc in X such that $p \in (a, b) \subset [a, b] \subset M$ and $M \cap P_X \subset \{p\}$. Suppose that a sequence $\{M_n\}_{n \in \mathbb{N}}$ in $C(X)$ converges, in the Hausdorff metric, to M. Then there is $N \in \mathbb{N}$ such that $p \in M_n$, for every $n > N$.*

Proof. Let us assume that, for infinitely many indices n, we have $M_n \cap (P_X - \{p\}) \neq \emptyset$. Since P_X is finite, there exists $q \in P_X - \{p\}$ such that $q \in M_n$, for infinitely many indices n. Thus, $q \in M$, so $M \cap P_X \not\subseteq \{p\}$. Since this is a contradiction, we have that $M_n \cap (P_X - \{p\}) \neq \emptyset$, only for finitely many indices n. Taking a subsequence of $\{M_n\}_{n \in \mathbb{N}}$, if necessary, we can consider that $M_n \cap P_X \subset \{p\}$, for every $n \in \mathbb{N}$.

Assume first that $p \in P_X$. If for infinitely many indices n, we have $M_n \cap P_X = \emptyset$, then $M \cap P_X = \emptyset$. Since this is a contradiction, there exists $N \in \mathbb{N}$ such that $M_n \cap P_X \neq \emptyset$, for each $n > N$. Thus, $p \in M_n$, for every $n > N$.

Assume now that $p \notin P_X$. Then $M \cap P_X = \emptyset$. If for infinitely many indices n, we have $M_n \cap P_X \neq \emptyset$, then $M \cap P_X \neq \emptyset$. This implies that $p \in P_X$, which is a contradiction. Hence, $M_n \cap P_X \neq \emptyset$, only for finitely many indices n. Taking a subsequence of $\{M_n\}_{n \in \mathbb{N}}$, if necessary, we can consider that $M_n \cap P_X = \emptyset$, for every $n \in \mathbb{N}$. Since $M \cap P_X = \emptyset$, there is $i \in \{1, 2, \ldots, l\}$ such that $M \subset D_i - P_X$. This implies that $M_n \subset D_i - P_X$ for all n, except finitely many of then. Taking again a subsequence of $\{M_n\}_{n \in \mathbb{N}}$, if necessary, we can assume that $M_n \subset D_i$ for all $n \in \mathbb{N}$. Since D_i is a dendrite, by Theorem 2.5, there exists $N \in \mathbb{N}$ such that $p \in M_n$, for every $n > N$. □

The following result is the equivalent version of [15, Lemma 4, p. 802] for the elements of $\mathfrak{LD}$. It says that, for $X \in \mathfrak{LD}$, arbitrarily close to X we can find a subset of X which is a finite graph.

Theorem 3.19. *Let $X \in \mathfrak{LD}$. Given $\varepsilon > 0$, there is $G \in C(X)$ such that $H(G, X) < \varepsilon$ and $G \cap E_a(X) = \emptyset$.*

Proof. Since $X = D_1 \cup D_2 \cup \cdots \cup D_l$ and, for each $i \in \{1, 2, \ldots, l\}$, we have $D_i \in \mathfrak{D}$, by [15, Lemma 4, p. 802], there is $G_i \in C(D_i)$ such that $H(D_i, G_i) < \varepsilon$ and $G_i \cap E_a(D_i) = \emptyset$. Note that $G_i \cap E_a(X) = \emptyset$. Let $i, j \in \{1, 2, \ldots, l\}$ be so that $i < j$ and

$D_i \cap D_j \neq \emptyset$. Fix $p_i \in G_i$, $p_j \in G_j$ and an arc A_{ij}, with end points p_i and p_j, such that $A_{ij} \subset D_i \cup D_j$. Since $p_i, p_j \notin E_a(X)$, we have $A_{ij} \cap E_a(X) = \emptyset$. Note that $G_i \cup G_j \cup A_{ij}$ is a subcontinuum of X such that

$$H(D_i \cup D_j, G_i \cup G_j \cup A_{ij}) < \varepsilon \quad \text{and} \quad (G_i \cup G_j \cup A_{ij}) \cap E_a(X) = \emptyset.$$

Let

$$G = \left(\bigcup_{i=1}^{l} G_i \right) \cup \left\{ A_{ij} \colon i, j \in \{1, 2, \ldots, l\},\ i < j \text{ and } D_i \cap D_j \neq \emptyset \right\}.$$

Then $G \in C(X)$, $H(G, X) < \varepsilon$ and $G \cap E_a(X) = \emptyset$. □

The following theorem characterizes the elements of the class $\mathfrak{LD}$. It is the equivalent version of [15, Theorem 8, p. 802] for the elements of $\mathfrak{LD}$.

Theorem 3.20. *Suppose $X \in \mathfrak{L}$. Then $X \in \mathfrak{LD}$ if and only if X satisfies the following property*:

($\star$) *for each $Z \in C(X)$ there is a sequence $\{A_n\}_{n \in \mathbb{N}}$ in $C(X)$, whose limit in the Hausdorff metric is Z, and $\dim_{A_n}(C(X)) < \infty$ for each $n \in \mathbb{N}$.*

Proof. Assume first that $X \in \mathfrak{LD}$ and let $Z \in C(X)$. By Corollary 3.6, $Z \in \mathfrak{LD}$. Given $n \in \mathbb{N}$, by Theorem 3.19, there is $B_n \in C(Z)$ such that $H(B_n, Z) < \frac{1}{2n}$ and $B_n \cap E_a(Z) = \emptyset$. By part (b) of Corollary 3.14, B_n is a finite graph contained in Z, so there is a subcontinuum A_n of B_n such that $A_n \cap E_a(X) = \emptyset$ and $H(A_n, B_n) < \frac{1}{2n}$. By Corollary 3.15, A_n has no essential points of X so, by Theorem 3.16, $\dim_{A_n}(C(X)) < \infty$. Hence, X satisfies ($\star$).

Assume now that X satisfies ($\star$) and that $X \notin \mathfrak{LD}$. We claim that:

1) there exist a nondegenerate arc $[a, b]$ in X and an essential point p of X such that $p \in (a, b) \subset [a, b]$, $[a, b] \cap P_X \subset \{p\}$ and $\dim_{[a,b]}(C(X)) = \infty$.

To show this note that, by Theorem 3.4, X contains either a copy of W or a copy of F_ω (defined in (2.1) and (2.2)). Assume first that X contains a copy of W. Then there exist a nondegenerate arc $[a, b]$ in X and a point $p \in (a, b) \subset [a, b]$ such that p is the limit of a sequence of different ramification points of X, all in the subarc $(p, b]$ of $[a, b]$. If $p \notin P_X$, then we can consider the arc $[a, b]$ so that $[a, b] \cap P_X = \emptyset$. If $p \in P_X$ then, since P_X is finite, we can assume that $[a, b]$ is such that $[a, b] \cap P_X = \{p\}$. In each case it follows that p is a *II*-essential point of X such that $p \in [a, b]$ so, by Theorem 2.7, $\dim_{[a,b]}(C(X)) = \infty$.

Assume now that X contains a copy F of F_ω with vertex p. Then $\operatorname{ord}(p, X) = \infty$, so p is a *I*-essential point of X. Let us consider that $p \notin P_X$. Since P_X is finite, we can find a copy F_0 of F_ω, with vertex p, such that $F_0 \subset F$ and $F_0 \cap P_X = \emptyset$. Let $[a, b]$ be a nondegenerate arc in F_0 such that $p \in (a, b) \subset [a, b]$. Then $[a, b] \cap P_X = \emptyset$. If $p \in P_X$ then, since P_X is finite, there exists a nondegenerate arc $[a, b]$ in F such that $p \in (a, b) \subset [a, b]$ and $[a, b] \cap P_X = \{p\}$. In each case, since p is an essential point of X such that $p \in [a, b]$, by Theorem 2.7, $\dim_{[a,b]}(C(X)) = \infty$. This completes the proof of 1).

Let $[a, b]$ be a nondegenerate arc in X that satisfies 1). Since X satisfies ($\star$), there is a sequence $\{M_n\}_{n \in \mathbb{N}}$ in $C(X)$, whose limit in the Hausdorff metric is $[a, b]$, and $\dim_{M_n}(C(X)) < \infty$ for each $n \in \mathbb{N}$. By Theorem 2.7, for every $n \in \mathbb{N}$ and each $q \in M_n$, q is not an essential point of X. Thus, $p \notin M_n$ for every $n \in \mathbb{N}$. However, by Theorem 3.18, there is $N \in \mathbb{N}$ such that $p \in M_n$ for every $n > N$. This contradiction shows that $X \in \mathfrak{LD}$. □

As an application of Theorem 3.20, we present the following result, which is an important part of the proof of the main theorem of this paper.

Theorem 3.21. *Let $X \in \mathfrak{LD}$ and $Y \in \mathfrak{L}$ be such that $C(X)$ is homeomorphic to $C(Y)$. Then $Y \in \mathfrak{LD}$.*

Proof. Let $h : C(X) \to C(Y)$ be a homeomorphism. Let $Z \in C(Y)$ and $A \in C(X)$ be so that $h(A) = Z$. Since $X \in \mathfrak{LD}$, by Theorem 3.20, there is a sequence $\{A_n\}_{n \in \mathbb{N}}$ in $C(X)$, whose limit in the Hausdorff metric is A, and $\dim_{A_n}(C(X)) < \infty$ for each $n \in \mathbb{N}$. Given $n \in \mathbb{N}$, let $Z_n = h(A_n)$. Then $\{Z_n\}_{n \in \mathbb{N}}$ is a sequence in $C(Y)$, whose limit in the Hausdorff metric is $h(A) = Z$, and $\dim_{Z_n}(C(Y)) < \infty$ for every $n \in \mathbb{N}$. This implies that Y satisfies ($\star$) of Theorem 3.20 so, by the same theorem, $Y \in \mathfrak{LD}$. □

4. The class $\boldsymbol{\Omega(X)}$

We recall that if V is a 2-cell, then ∂V represents the manifold boundary of V. Given a continuum X, we consider the following subset of $C(X)$:

$$\Omega(X) = \{A \in C(X)\colon \text{ there exists a 2-cell } \mathcal{V} \text{ in } C(X) \text{ such that } A \in \mathrm{Int}_{C(X)}(\mathcal{V}) \cap \partial\mathcal{V}\}.$$

In connection to the problem of finding conditions on a continuum X, in order that X has unique hyperspace, the class $\Omega(X)$ plays the role described in the following result. Such result was proved in [15, Lemma 2, p. 801] for the elements of $\mathfrak{D}$, but it is valid in general, with the same proof.

Theorem 4.1. *Let X and Y be continua such that $C(X)$ is homeomorphic to $C(Y)$. Then $\Omega(X)$ is homeomorphic to $\Omega(Y)$.*

In this section we will prove that if $X \in \mathfrak{LD}$ is different from an arc, then $\mathrm{Cl}_{C(X)}(\Omega(X))$ is homeomorphic to X (see Theorem 4.10). To do this we require some previous results. The first two are proved in [15, Proposition 2, p. 799] and [15, Lemma 6, p. 803], respectively.

Theorem 4.2. *Let X be a continuum. If $A \in \Omega(X)$, then $\dim_A(C(X)) \leqslant 2$.*

Theorem 4.3. *Let X be a locally connected continuum and $p \in X$. If $\dim_{\{p\}}(C(X)) < \infty$, then $\{p\} \in \mathrm{Cl}_{C(X)}(\Omega(X))$.*

Theorem 4.4. *Let X be a locally connected continuum that satisfies $(\star)$ of Theorem 3.20. Let Y be a continuum such that $C(X)$ is homeomorphic to $C(Y)$. Then $F_1(Y) \subset \mathrm{Cl}_{C(Y)}(\Omega(Y))$.*

Proof. By Theorem 2.2, Y is locally connected. Let $h : C(X) \to C(Y)$ be a homeomorphism, $\{p\} \in F_1(Y)$ and $A \in C(X)$ be such that $h(A) = \{p\}$. By $(\star)$ of Theorem 3.20, there is a sequence $\{A_n\}_{n\in\mathbb{N}}$ in $C(X)$, whose limit in the Hausdorff metric is A, and $\dim_{A_n}(C(X)) < \infty$ for each $n \in \mathbb{N}$. Then $\{h(A_n)\}_{n\in\mathbb{N}}$ is a sequence in $C(Y)$, whose limit in the Hausdorff metric is $h(A) = \{p\}$, and $\dim_{h(A_n)}(C(Y)) < \infty$ for each $n \in \mathbb{N}$. Let $\{p_n\}_{n\in\mathbb{N}}$ be a convergent sequence in Y, whose limit is p, and $p_n \in h(A_n)$ for every $n \in \mathbb{N}$. For all $n \in \mathbb{N}$, since Y is locally connected, $p_n \in h(A_n)$ and $\dim_{h(A_n)}(C(Y)) < \infty$, by Theorem 2.1, we have $\dim_{\{p_n\}}(C(Y)) < \infty$ and then, by Theorem 4.3, $\{p_n\} \in \mathrm{Cl}_{C(Y)}(\Omega(Y))$. Thus, $\{p\} \in \mathrm{Cl}_{C(Y)}(\Omega(Y))$. $\square$

Combining Theorems 3.20 and 4.4, we obtain the following result which is the equivalent version of [15, Lemma 7, p. 803], for the elements of $\mathfrak{LD}$.

Theorem 4.5. *Let $X \in \mathfrak{LD}$ and let Y be a continuum such that $C(X)$ is homeomorphic to $C(Y)$. Then $F_1(Y) \subset \mathrm{Cl}_{C(Y)}(\Omega(Y))$.*

Theorem 4.6. *Let $X \in \mathfrak{LD}$ and $A \in \Omega(X)$. Then*

$$A \cap E_a(X) = \emptyset \quad \textit{and} \quad A \cap R(X) = \emptyset.$$

Proof. Let $A \in \Omega(X)$. If $A \cap E_a(X) \neq \emptyset$ then, by Corollary 3.15, there exists $p \in A$ such that p is an essential point of X. Then, by Theorem 3.16, $\dim_A(C(X)) = \infty$. Since this contradicts Theorem 4.2, we obtain that $A \cap E_a(X) = \emptyset$.

Now assume that there is $p \in A \cap R(X)$. Since $A \in \Omega(X)$, there exists a 2-cell $\mathcal{V}$ in $C(X)$ such that $A \in \mathrm{Int}_{C(X)}(\mathcal{V}) \cap \partial\mathcal{V}$. Let $\varepsilon > 0$ be such that $B_{C(X)}(A, \varepsilon) \subset \mathcal{V}$. Since $3 \leqslant \mathrm{ord}(p, X) < \infty$, there exist $n \in \mathbb{N} - \{1, 2\}$ and an n-od T in X such that $T \in B_{C(X)}(A, \frac{\varepsilon}{2})$. By Theorem 2.4, there is an n-cell Γ such that $T \in \Gamma \subset B_{C(X)}(A, \varepsilon)$. This implies that the 2-cell $\mathcal{V}$ contains the n-cell Γ, a contradiction. Therefore, $A \cap R(X) = \emptyset$. $\square$

Theorem 4.7. *Let X be a continuum, $A \in C(X) - F_1(X)$ and $[p, q]$ be a nondegenerate arc in X, with end points p and q, such that $(p, q) = [p, q] - \{p, q\}$ is open in X and $A \subset (p, q)$. Then $A \notin \Omega(X)$.*

Proof. Let us assume that $A \in \Omega(X)$. Since A is a nondegenerate subcontinuum of an arc, there exist $a, b \in A$ such that $a \neq b$ and $A = [a, b]$. We can consider that, in the natural order $\leqslant$ of $[p, q]$ from p to q, we have $p < a < b < q$. Since $A \in \Omega(X)$, there exists a 2-cell $\mathcal{V}$ in $C(X)$ such that $A \in \mathrm{Int}_{C(X)}(\mathcal{V}) \cap \partial\mathcal{V}$. Let $\delta > 0$ be so that $B_{C(X)}(A, \delta) \subset \mathcal{V}$. Fix $c_1, c_2, d_1, d_2 \in [p, q]$ so that

$$p \leqslant c_1 < a < c_2 < d_2 < b < d_1 \leqslant q,$$

$[c_1, d_1] \in B_{C(X)}(A, \delta)$ and $[c_2, d_2] \in B_{C(X)}(A, \delta)$. It is not difficult to prove that

$$\mathcal{D} = \{[c, d] \subset [p, q]\colon c_1 \leqslant c \leqslant c_2 \text{ and } d_2 \leqslant d \leqslant d_1\}$$

is a 2-cell in $C(X)$ such that

$$A \in \mathcal{D} - \partial\mathcal{D} \subset \mathcal{D} \subset B_{C(X)}(A, \delta) \subset \mathcal{V}.$$

Since both $\mathcal{D}$ and $\mathcal{V}$ are 2-cells and $\mathcal{D} \subset \mathcal{V}$, the set $\mathcal{D} - \partial\mathcal{D}$ is open in $\mathcal{V}$ [37, 19.34, p. 123]. Thus, A is in the manifold interior of $\mathcal{V}$, so $A \notin \partial\mathcal{V}$. Since this is a contradiction, we conclude that $A \notin \Omega(X)$. $\square$

For a continuum Y, a *free arc in Y* is an arc A in Y, joining two different points p and q of Y, such that the set $A-\{p,q\}$ is open in Y. We recall that if $p\in Y$, then $C(p,Y)=\{A\in C(Y)\colon\ p\in A\}$. If A is an arc in Y, joining two different points p and q, and B is a simple closed curve in Y, then both $C(A)$ and $C(B)$ are 2-cells in $C(Y)$. Using Theorem 4.7, it follows that

$$\Omega(A)=\partial C(A)=C(p,A)\cup C(q,A)\cup F_1(A)\quad\text{and}\quad \Omega(B)=F_1(B).$$

Note that $\Omega(A)$ is homeomorphic to $\Omega(B)$.

Theorem 4.8. *Let $X\in\mathfrak{LD}$ be different from an arc and $A\in C(X)$. Then $A\in\Omega(X)$ if and only if A satisfies exactly one of the following conditions*:

(a) $A=\{p\}$*, for some* $p\in X-(R(X)\cup E_a(X))$;
(b) *A is nondegenerate arc $[e,p]$ in X, such that $e\in E(X)-E_a(X)$ and $(e,p]\subset O(X)$.*

Moreover $A\in \mathrm{Cl}_{C(X)}(\Omega(X))-\Omega(X)$ if and only if A satisfies exactly one of the following conditions:

(c) $A=\{r\}$*, for some* $r\in R(X)\cup E_a(X)$;
(d) $A=[e,r]$*, for some $e\in E(X)-E_a(X)$ and $r\in R(X)$ so that $(e,r)\subset O(X)$.*

Proof. If X is a simple closed curve, then

$$\Omega(X)=F_1(X)=\mathrm{Cl}_{C(X)}\big(\Omega(X)\big).$$

Thus, in this situation, the result is true. Consider then that X is not a simple closed curve. To show the first part of the theorem, assume that $A\in\Omega(X)$. By Theorem 3.11, we have

$$X=E_a(X)\cup\big(E(X)-E_a(X)\big)\cup O(X)\cup R(X).$$

By Theorem 4.6, $A\cap E_a(X)=\emptyset=A\cap R(X)$. Thus, $A\subset(E(X)-E_a(X))\cup O(X)$. Since $A\cap E_a(X)=\emptyset$, by part (b) of Corollary 3.14, A is a finite graph in X and, since $A\cap R(X)=\emptyset$, either A is a one-point-set or a nondegenerate arc in X without ramification points of X. Assume first that $A=\{p\}$, for some $p\in X$. Then $p\in(E(X)-E_a(X))\cup O(X)$, so A satisfies (a).

Assume now that A is a nondegenerate arc $[a,b]$ in X. Consider that $A\subset O(X)$. By Corollary 3.13, $O(X)$ is open in X. Hence, there is an arc $[p,q]$ in $O(X)$ such that $(p,q)=[p,q]-\{p,q\}$ is open in X and $A\subset(p,q)$. Then, by Theorem 4.7, $A\notin\Omega(X)$. Since this is a contradiction and $A\subset(E(X)-E_a(X))\cup O(X)$, it follows that $A\cap(E(X)-E_a(X))\neq\emptyset$. Since X is not an arc, A is of the form described in (b).

Let us assume now that $A\in C(X)$ satisfies (a). Then $A=\{p\}$, for some $p\in(E(X)-E_a(X))\cup O(X)$. If $p\in O(X)$ then, since $O(X)$ is open in X, there is a nondegenerate free arc $[a,b]$ in $O(X)$ such that $p\in(a,b)\subset[a,b]$. Let $\mathcal{V}=C([a,b])$. Then $\mathcal{V}$ is a 2-cell in $C(X)$ such that $A\in\partial\mathcal{V}$. Let $\varepsilon>0$ be such that $B_X(p,\varepsilon)\subset(a,b)$. It is easy to see that $B_{C(X)}(A,\varepsilon)\subset\mathcal{V}$, so $A\in\mathrm{Int}_{C(X)}(\mathcal{V})$. Hence, $A\in\Omega(X)$.

If $p\in E(X)-E_a(X)$, then there exists $t\in X$ such that $(p,t]\subset O(X)$. Then $[p,t]$ is a free arc in X. Let $\mathcal{V}=C([p,t])$. Then $\mathcal{V}$ is a 2-cell in $C(X)$ such that $A\in\partial\mathcal{V}$. Let $\varepsilon>0$ be such that $B_X(p,\varepsilon)\subset(p,t]$. It is easy to see that $B_{C(X)}(A,\varepsilon)\subset\mathcal{V}$, so $A\in\mathrm{Int}_{C(X)}(\mathcal{V})$. Hence, $A\in\Omega(X)$.

Assume now that $A\in C(X)$ satisfies (b). Then $A=[e,p]$, for some $e\in E(X)-E_a(X)$ and $(e,p]\subset O(X)$. Since $O(X)$ is open in X, there exists $q\in X$ such that $A\subset[e,q)\subset[e,q]$ and $(e,q]\subset O(X)$. Then $[e,q]$ is a free arc in X. Let $\mathcal{V}=C([e,q])$. Then $\mathcal{V}$ is a 2-cell in $C(X)$ and, since $A\in C(e,[e,q])$, we have $A\in\partial\mathcal{V}$. Let $\varepsilon>0$ be such that $N(\varepsilon,A)\subset[e,q)$. It is easy to see that $B_{C(X)}(A,\varepsilon)\subset\mathcal{V}$, so $A\in\mathrm{Int}_{C(X)}(\mathcal{V})$. Hence, $A\in\Omega(X)$. This concludes the proof of the first part of the theorem.

To show the second part of the theorem, assume that

$$A\in\mathrm{Cl}_{C(X)}\big(\Omega(X)\big)-\Omega(X).$$

Then there is a sequence $\{A_n\}_{n\in\mathbb{N}}$ in $\Omega(X)$, whose limit in the Hausdorff metric is A. If for infinitely many indices n we have $A_n\in F_1(X)$, then $A\in F_1(X)$. Thus, $A=\{p\}$ for some $p\in X$. Since $A\notin\Omega(X)$, by (a), we have $p\in R(X)\cup E_a(X)$. Hence, A satisfies (c). Assume now that $A_n\in F_1(X)$ only for finitely many indices n. Taking a subsequence of $\{A_n\}_{n\in\mathbb{N}}$, if necessary, we can consider that $A_n\notin F_1(X)$, for every $n\in\mathbb{N}$. Thus, given $n\in\mathbb{N}$, by (b), $A_n=[e_n,p_n]$ for some $e_n\in E(X)-E_a(X)$ and $(e_n,p_n]\subset O(X)$. We claim that:

1) if $Y\in C(X)\cap\mathfrak{D}$ is such that $A_n\subset Y$, for infinitely many indices n, then A satisfies either (c) or (d).

Let Y be as assumed. Taking a subsequence of $\{A_n\}_{n\in\mathbb{N}}$, if necessary, we can consider that $A_n\subset Y$ for every $n\in\mathbb{N}$. Then $A\subset Y$. Note that $e_n\in E(Y)-E_a(Y)$ and $(e_n,p_n)\subset O(Y)$, for every $n\in\mathbb{N}$. Taking subsequences of $\{e_n\}_{n\in\mathbb{N}}$ and $\{p_n\}_{n\in\mathbb{N}}$, if necessary, we can also assume that $e_n\to e$ and $p_n\to p$, for some $e\in E(Y)$ and $p\in Y$. Let us consider that the set

$\{e_n\colon n \in \mathbb{N}\} - \{e\}$ has infinitely many elements. Then $e \in E_a(Y)$, so $e \in E_a(X)$. Since $Y \in \mathfrak{D}$, by Theorem 3.2, there exist an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ in $\mathbb{N}$ so that $e_{n_k} \neq e$, for every $k \in \mathbb{N}$, and a sequence $\{r_{n_k}\}_{k\in\mathbb{N}}$ of distinct ramification points of Y, all in the nondegenerate arc $[e, e_{n_1}]$, such that e is the limit of $\{r_{n_k}\}_{k\in\mathbb{N}}$ and

$$T = [e, r_{n_1}] \cup \left(\bigcup_{k\in\mathbb{N}} [e_{n_k}, r_{n_k}] \right)$$

is a subcontinuum of Y, homeomorphic to the dendrite W_0 defined in (2.3). Since $A_{n_k} = [e_{n_k}, p_{n_k}]$ does not have ramification points of X, it follows that $p_{n_k} \in (e_{n_k}, r_{n_k})$ for every $k \in \mathbb{N}$. Since the sequence $\{e_{n_k}\}_{k\in\mathbb{N}}$ converges to e, by property (S) described on p. 2072, the sequence of arcs $\{[e, e_{n_k}]\}_{k\in\mathbb{N}}$ converges, in the Hausdorff metric, to $\{e\}$. Since $[e_{n_k}, p_{n_k}] \subset [e, e_{n_k}]$, for every $k \in \mathbb{N}$, it follows that the sequence of arcs $\{[e_{n_k}, p_{n_k}]\}_{k\in\mathbb{N}}$ converges, in the Hausdorff metric, to $\{e\}$. Since the sequence $\{[e_{n_k}, p_{n_k}]\}_{k\in\mathbb{N}}$ also converges, in the Hausdorff metric, to A, we conclude that $A = \{e\}$, so A satisfies (c).

Let us assume now that the set $\{e_n\colon n \in \mathbb{N}\} - \{e\}$ is finite. Then there is $N \in \mathbb{N}$ such that $e_n = e$, for every $n \geqslant N$. Hence, $e \in E(X) - E_a(X)$ and $A_n = [e, p_n]$, for each $n \geqslant N$. Since the sequence $\{p_n\}_{n\geqslant N}$ converges to p, by property (S) described on p. 2072, the sequence of arcs $\{A_n\}_{n\geqslant N}$ converges, in the Hausdorff metric, to $[e, p]$. This implies that $A = [e, p]$. If $p = e$, then A satisfies (a) so $A \in \Omega(X)$. This is a contradiction, so $p \neq e$. Since $(e, p_n] \subset O(X)$, for every $n \geqslant N$, it follows that $(e, p) \subset O(X)$. If $p \in O(X)$, then A satisfies (b), so $A \in \Omega(X)$. This contradiction shows that $p \notin O(X)$. If $p \in E(X)$ then $X = Y = A$, so X is an arc. This is a contradiction, so $p \notin E(X)$. Hence, $p \in R(X)$ and then A satisfies (d). This completes the proof of 1).

Recall that, by Theorem 3.7, $X = D_1 \cup D_2 \cup \cdots \cup D_l$, where each $D_i \in \mathfrak{D}$ and $|D_i \cap D_j| < \infty$ for every $i, j \in \{1, 2, \ldots, l\}$ with $i \neq j$. Let $i_1 \in \{1, 2, \ldots, l\}$ be such that $e_n \in D_{i_1}$ for infinitely many indices n. Taking a subsequence of $\{A_n\}_{n\in\mathbb{N}}$, if necessary, we can consider that $e_n \in D_{i_1}$, for every $n \in \mathbb{N}$. If $A_n \subset D_{i_1}$ for infinitely many indices n then, by 1), A satisfies either (c) or (d).

Assume then that $A_n \subset D_{i_1}$ only for finitely many indices n. Taking a subsequence of $\{A_n\}_{n\in\mathbb{N}}$, if necessary, we can consider that $A_n \not\subseteq D_{i_1}$, for all $n \in \mathbb{N}$. Since $A_1 \not\subseteq D_{i_1}$, there exist $i_2 \in \{1, 2, \ldots, l\} - \{i_1\}$ and $q_1 \in D_{i_1} \cap D_{i_2}$ such that $A_1 = [e_1, q_1] \cup [q_1, p_1]$, $[e_1, q_1] \subset D_{i_1}$ and $[q_1, p_1] \cap (D_{i_2} - \{q_1\}) \neq \emptyset$. We claim that:

2) $D_{i_1} = [e_1, q_1]$ and $e_n = e_1$, for every $n \in \mathbb{N}$.

To show 2) note that, since $e_1 \in E(X)$ and $q_1 \in P_X \subset X - E(X)$, we have $e_1 \neq q_1$. Moreover $e_1 \in E(D_{i_1})$ and, since $q_1 \in O(X)$, it follows that $q_1 \in E(D_{i_1}) \cap E(D_{i_2})$. If there exists $t \in D_{i_1} - [e_1, q_1]$ then, since D_{i_1} is arcwise connected and $e_1, q_1 \in E(D_{i_1})$, any arc in D_{i_1} from t to a point in $[e_1, q_1]$ intersects (e_1, q_1). Then, some point of (e_1, q_1) is a ramification point of D_{i_1}. This implies that some point of $(e_1, p_1]$ is a ramification point of X. This contradicts the fact that $(e_1, p_1] \subset O(X)$, so $D_{i_1} = [e_1, q_1]$. Now let $n \in \mathbb{N}$. Since $e_n \in E(X) \cap D_{i_1}$ and $q_1 \notin E(X)$, we have $e_n \in E(D_{i_1}) - \{q_1\}$. Then $e = e_1$, since $D_{i_1} = [e_1, q_1]$. This shows 2).

Now we claim that:

3) $D_{i_1} \cap D_{i_2} = \{q_1\}$.

To show 3) assume, on the contrary, that there exists $t \in D_{i_1} \cap D_{i_2}$ such that $t \neq q_1$. Then $t \in D_{i_2} \cap P_X$ so, by Theorem 3.9, there is a nondegenerate arc C in D_{i_2} such that $C \cap P_X = \{t\}$. Thus, $C \cap D_{i_1} = \{t\}$. Since $D_{i_1} = [e_1, q_1]$ and $C \cap D_{i_1} = \{t\}$, if $t = e_1$, then $e_1 \notin E(X)$ and, if $t \in (e_1, q_1)$, then $t \in R(X)$. In the first case we contradict the fact that $e_1 \in E(X)$ and, in the second case, the fact that $(e_1, q_1] \subset O(X)$. Hence, $D_{i_1} \cap D_{i_2} = \{q_1\}$. This shows 3).

Now we are going to prove that:

4) there is no $i \in \{1, 2, \ldots, l\} - \{i_1, i_2\}$ such that $q_1 \in D_{i_1} \cap D_{i_2} \cap D_i$, and there is no $i \in \{1, 2, \ldots, l\} - \{i_1\}$ such that $(D_{i_1} - \{q_1\}) \cap D_i \neq \emptyset$.

To show the first part of 4), let $i \in \{1, 2, \ldots, l\} - \{i_1, i_2\}$ be such that $q_1 \in D_{i_1} \cap D_{i_2} \cap D_i$. Then $q_1 \in D_i \cap P_X$ so, by Theorem 3.9, there is a nondegenerate arc D in D_i such that $D \cap P_X = \{q_1\}$. This implies that $q_1 \in R(X)$ and, since this contradicts the fact that $q_1 \in O(X)$, the first part of 4) is true. To show the second part of 4), let $i \in \{1, 2, \ldots, l\} - \{i_1\}$ be such that $(D_{i_1} - \{q_1\}) \cap D_i \neq \emptyset$. Let $t \in (D_{i_1} - \{q_1\}) \cap D_i$. Then $t \in P_X \subset X - E(X)$, so $t \neq e_1$. Since $t \in D_i \cap P_X$, by Theorem 3.9, there is a nondegenerate arc D' in D_i such that $D' \cap P_X = \{t\}$. This implies that $t \in R(X)$. Since this contradicts the fact that $(e_1, p_1] \subset O(X)$, the second part of 4) is true.

Now we claim that:

5) $D_{i_1} \subset A_n$, for every $n \in \mathbb{N}$.

To show 4) let $n \in \mathbb{N}$. Then, by 2), $A_n = [e_1, p_n]$. Since both A_n and $[e_1, q_1]$ are arcs with one common end point, $A_n \cap [e_1, q_1]$ is a subarc of $[e_1, q_1] = D_{i_1}$. If such subarc is proper then, since $A_n \not\subseteq D_{i_1}$, there is $i \in \{1, 2, \ldots, l\} - \{i_1\}$ such that $(D_{i_1} - \{q_1\}) \cap D_i \neq \emptyset$. This contradicts the second part of 4), so $A_n \cap [e_1, q_1] = [e_1, q_1]$, and then $D_{i_1} = [e_1, q_1] \subset A_n$. This shows 5).

Now we claim that:

6) $D_{i_1} \cup D_{i_2} \in C(X) \cap \mathfrak{D}$.

By 3), it follows that $D_{i_1} \cup D_{i_2} \in C(X)$. Since $q_1 \in O(X) \cap P_X$, we have $q_1 \in E(D_{i_2})$. By Theorem 3.10, $q_1 \notin E_a(D_{i_2})$, so $q_1 \in E(D_{i_2}) - E_a(D_{i_2})$ and, since D_{i_1} is an arc that intersects D_{i_2} only at q_1, $D_{i_1} \cup D_{i_2}$ is homeomorphic to D_{i_2}. Hence, $D_{i_1} \cup D_{i_2} \in \mathfrak{D}$. This shows 6).

If $A_n \subset D_{i_1} \cup D_{i_2}$ for infinitely many indices n then, by 1) and 6), A satisfies either (c) or (d). Assume then that $A_n \subset D_{i_1} \cup D_{i_2}$ only for finitely many indices n. Taking a subsequence of $\{A_n\}_{n\in\mathbb{N}}$, if necessary, we can consider that $A_n \nsubseteq D_{i_1} \cup D_{i_2}$, for all $n \in \mathbb{N}$. Since $A_1 \nsubseteq D_{i_1} \cup D_{i_2}$, there exist $i_3 \in \{1, 2, \dots, l\} - \{i_1, i_2\}$ and $q_2 \in D_{i_2} \cap D_{i_3}$ such that $A_1 = [e_1, q_1] \cup [q_1, q_2] \cup [q_2, p_1]$, $[q_1, q_2] \subset D_{i_2}$ and $[q_2, p_1] \cap (D_{i_3} - \{q_2\}) \neq \emptyset$. Note that $q_2 \neq q_1$. Proceeding as in the proof of 2)–6) we infer that $D_{i_2} = [q_1, q_2]$, $D_{i_3} \cap (D_{i_1} \cup D_{i_2}) = \{q_2\}$, $D_{i_1} \cup D_{i_2} \cup D_{i_3} \in C(X) \cap \mathfrak{D}$ and $D_{i_1} \cup D_{i_2} \subset A_n$, for every $n \in \mathbb{N}$. We also infer that there is no $i \in \{1, 2, \dots, l\} - \{i_1, i_2, i_3\}$ such that $q_2 \in D_{i_2} \cap D_{i_3} \cap D_i$ and that there is no $i \in \{1, 2, \dots, l\} - \{i_2\}$ such that $(D_{i_2} - \{q_1, q_2\}) \cap D_i \neq \emptyset$.

If we continue the above argument, we conclude that there exists $Y \in C(X) \cap \mathfrak{D}$ such that $A_n \subset Y$ for infinitely many indices n. By 1), this implies that A satisfies either (c) or (d). This shows that if $A \in \mathrm{Cl}_{C(X)}(\Omega(X)) - \Omega(X)$, then A satisfies either (c) or (d).

Now assume that $A \in C(X)$ satisfies either (c) or (d). Then A does not satisfy (a) nor (b) so, by the first part of the theorem, $A \notin \Omega(X)$. If A satisfies (c), then $A = \{r\}$, for some $r \in R(X) \cup E_a(X)$. By Theorem 3.17, there exists a sequence $\{p_n\}_{n\in\mathbb{N}}$ in $O(X)$ that converges to r. Then A is the limit, in the Hausdorff metric, of the sequence $\{\{p_n\}\}_{n\in\mathbb{N}}$ and, by the first part of the theorem, $\{p_n\} \in \Omega(X)$ for every $n \in \mathbb{N}$. Thus, $A \in \mathrm{Cl}_{C(X)}(\Omega(X)) - \Omega(X)$.

If A satisfies (d), then $A = [e, r]$ for some $e \in E(X) - E_a(X)$ and $r \in R(X)$ such that $(e, r) \subset O(X)$. Let $\{p_n\}_{n\in\mathbb{N}}$ be a sequence in (e, r) that converges to r. Given $n \in \mathbb{N}$ the subarc $[e, p_n]$ of $[e, r]$ satisfies (b) so, by the first part of the theorem, $[e, p_n] \in \Omega(X)$. By property (S) described on p. 2072, the sequence of arcs $\{[e, p_n]\}_{n\in\mathbb{N}}$ converges, in the Hausdorff metric, to $[e, r]$. Thus, $A \in \mathrm{Cl}_{C(X)}(\Omega(X)) - \Omega(X)$. □

Corollary 4.9. *If $X \in \mathfrak{LD}$, then $F_1(X) \subset \mathrm{Cl}_{C(X)}(\Omega(X))$.*

The following result is another important part of the proof of the main theorem of this paper. By Theorem 4.8, $\mathrm{Cl}_{C(X)}(\Omega(X))$ is the union of $F_1(X)$ with all the sets of the form $C(e, [e, r])$, where $[e, r]$ satisfies (d) of such theorem. Every such set $C(e, [e, r])$ is an arc in $C(X)$, with end points $\{e\}$ and $[e, r]$, that intersects $F_1(X)$ only at $\{e\}$. Moreover, if $[e_1, r_1]$ and $[e_2, r_2]$ are distinct elements of $C(X)$ that satisfy (d), then $C(e_1, [e_1, r_1]) \cap C(e_2, [e_2, r_2]) = \emptyset$. Then, in order to construct $\mathrm{Cl}_{C(X)}(\Omega(X))$ we attach to $F_1(X)$, one arc through every isolated end point of X, in such a way that the attached arcs are pairwise disjoint. We can then intuit that $\mathrm{Cl}_{C(X)}(\Omega(X))$ is homeomorphic to X. This will be the case if we prove that if $\{[e_n, r_n]\}_{n\in\mathbb{N}}$ is a sequence of elements in $C(X)$ that satisfy (d) and $e \in X$ is the limit of the sequence $\{e_n\}_{n\in\mathbb{N}}$, then the sequence $\{C(e_n, [e_n, r_n])\}_{n\in\mathbb{N}}$ converges, in the Hausdorff metric, to $\{\{e\}\}$. Though this is the case, we will show that $\mathrm{Cl}_{C(X)}(\Omega(X))$ is homeomorphic to X, using a different argument.

Theorem 4.10. *If $X \in \mathfrak{LD}$ is different from an arc, then $\mathrm{Cl}_{C(X)}(\Omega(X))$ is homeomorphic to X.*

Proof. Since the result is true when X is a simple closed curve, assume that X is not a simple closed curve. We will first define a one-to-one function $g : \Omega(X) \to X$ which we will extend to a homeomorphism $G : \mathrm{Cl}_{C(X)}(\Omega(X)) \to X$. To define g let us consider the following class of arcs in X:

$$\mathcal{E} = \big\{[e, r]\colon\ e \in E(X) - E_a(X),\ r \in R(X) \text{ and } (e, r) \subset O(X)\big\}.$$

Since X is different from an arc and a simple closed curve, for every $e \in E(X) - E_a(X)$ there is a unique $r_e \in R(X)$ such that $[e, r_e] \in \mathcal{E}$. For every such $[e, r_e]$, fix a point $s_e \in (e, r_e)$ as well as two homeomorphisms $f_1^{s_e} : [e, r_e] \to [s_e, r_e]$ and $f_2^{s_e} : [e, r_e] \to [e, s_e]$ so that

$$f_1^{s_e}(e) = s_e, \qquad f_1^{s_e}(r_e) = r_e, \qquad f_2^{s_e}(e) = s_e \quad \text{and} \quad f_2^{s_e}(r_e) = e.$$

Note that the sets (e, s_e) and $[s_e, r_e]$ are disjoint. We are ready to define g. Let $A \in \Omega(X)$. By Theorem 4.8, A satisfies either (a) or (b) of such theorem. Assume first that $A = \{a\}$, for some $a \in O(X)$ and that, for every $[e, r] \in \mathcal{E}$, we have $a \notin [e, r]$. Define

$$g(A) = g\big(\{a\}\big) = a.$$

Let us assume now that $A = \{q\}$, for some $q \in (E(X) - E_a(X)) \cup O(X)$ so that $q \in [e, r_e]$, for some $[e, r_e] \in \mathcal{E}$. Define

$$g(A) = g(\{q\}) = f_1^{s_e}(q).$$

Note that $g(A) \in [s_e, r_e) \subset [e, r_e]$, so $g(A)$ is an element of some member of $\mathcal{E}$. Note also that we have defined g for every member of $\Omega(X)$ that satisfies (a) of Theorem 4.8. Now assume that A satisfies (b) of Theorem 4.8. Then there exists $[e, r_e] \in \mathcal{E}$ and $p \in (e, r_e)$ such that $A = [e, p] \subset [e, r_e]$. Define

$$g(A) = g([e, p]) = f_2^{s_e}(p).$$

Note that $g(A) \in (e, s_e) \subset [e, r_e]$, so $g(A)$ is an element of some member of $\mathcal{E}$. Note also that g is a well-define function so that $g(A) \in O(X)$, for every $A \in \Omega(X)$. We claim that:

1) g is a one-to-one function.

To show 1) let $A, D \in \Omega(X)$ be such that $g(A) = g(D)$. Assume first that $A = \{a\}$, for some $a \in O(X)$ and that, for every $[e, r] \in \mathcal{E}$, we have $a \notin [e, r]$. Then $g(A) = a$, so $g(D) = a$. This implies that D is an element of $\Omega(X)$ so that $g(D)$ does not belong to any member of $\mathcal{E}$. Then, by the way g is defined, it follows that $D = \{a\} = A$.

Now assume that $A = \{q\}$, for some $q \in (E(X) - E_a(X)) \cup O(X)$ so that $q \in [e, r_e]$ and $[e, r_e] \in \mathcal{E}$. Then $g(A) = f_1^{s_e}(q)$, so $g(D) = f_1^{s_e}(q) \in [s_e, r_e) \subset [e, r_e)$. This implies that D is not of the form of A described in the previous paragraph and also that $D \subset [e, r_e)$. If $D = [e, p]$ for some $p \in (e, r_e)$, then $g(D) = f_2^{s_e}(p) \in (e, s_e)$. This contradicts the fact that $g(D) \in [s_e, r_e)$. Then $D = \{r\}$, for some $r \in (E(X) - E_a(X)) \cup O(X)$ so that $r \in [e, r_e]$. Hence, $g(D) = f_1^{s_e}(r)$, so $f_1^{s_e}(q) = f_1^{s_e}(r)$ and, since $f_1^{s_e}$ is a one-to-one function, $q = r$. Thus, $D = A$.

Now assume that there exists $[e, r_e] \in \mathcal{E}$ and $p \in (e, r_e)$ such that $A = [e, p] \subset [e, r_e]$. Then $g(A) = f_2^{s_e}(p)$, so $g(D) = f_2^{s_e}(p) \in (e, s_e) \subset [e, r_e]$. This is implies that D is not of the form of A described in the first paragraph of the proof of 1), and also that $D \subset [e, r_e]$. If $D = \{q\}$, for some $q \in (E(X) - E_a(X)) \cup O(X)$ so that $q \in [e, r_e]$, then $g(D) = f_1^{s_e}(q) \in [s_e, r_e)$. This contradicts the fact that $g(D) \in (e, s_e)$, so $D = [e, r]$ for some $r \in (e, r_e)$. Then $g(D) = f_2^{s_e}(r)$, so $f_2^{s_e}(p) = f_2^{s_e}(r)$ and, since $f_2^{s_e}$ is a one-to-one function, we have $p = r$. Thus, $D = A$. This completes the proof of 1).

We now define G. Let $A \in \mathrm{Cl}_{C(X)}(\Omega(X))$. If $A \in \Omega(X)$, then $G(A) = g(A)$. Assume now that $A \in \mathrm{Cl}_{C(X)}(\Omega(X)) - \Omega(X)$. Then A satisfies either (c) or (d) of Theorem 4.8. If A satisfies (c), then $A = \{s\}$, for some $s \in R(X) \cup E_a(X)$. Define

$$G(A) = G(\{s\}) = s.$$

If A satisfies (d), then $A = [e, r_e]$, for some $[e, r_e] \in \mathcal{E}$. Define

$$G(A) = G([e, r_e]) = e.$$

The function G is well defined and $G(A) \in X - O(X)$, for every $A \in \mathrm{Cl}_{C(X)}(\Omega(X)) - \Omega(X)$. Note that:

2) if $A \in \mathrm{Cl}_{C(X)}(\Omega(X)) - \Omega(X)$, then $G(A) \in R(X) \cup E_a(X)$ if A satisfies (c) of Theorem 4.8, while $G(A) \in E(X) - E_a(X)$, if A satisfies (d) of Theorem 4.8.

We claim that:

3) G is a one-to-one function.

To show 3) note that since G extends g, by 1), G is one-to-one in $\Omega(X)$. Note also that G extends g, $g(A) \in O(X)$ for each $A \in \Omega(X)$ and $G(B) \in X - O(X)$, for every $B \in \mathrm{Cl}_{C(X)}(\Omega(X)) - \Omega(X)$. According to this and 2), to complete the proof of 3), it is enough to show that if $A, B \in \mathrm{Cl}_{C(X)}(\Omega(X)) - \Omega(X)$ are such that $G(A) = G(B)$ and either both A and B satisfy (c) of Theorem 4.8, or both A and B satisfy (d) of Theorem 4.8, then $A = B$. So let A and B be as assumed. If both A and B satisfy (c), then $A = \{s\}$ and $B = \{t\}$ for some $s, t \in R(X) \cup E_a(X)$. Thus, $s = G(A) = G(B) = t$, so $A = B$. If both A and B satisfy (d), then $A = [e_1, r_1]$ and $B = [e_2, r_2]$, for some $[e_1, r_1], [e_2, r_2] \in \mathcal{E}$. Then $e_1 = G(A) = G(B) = e_2$ and, since for every $e \in E(X) - E_a(X)$ there is a unique $r_e \in R(X)$ such that $[e, r_e] \in \mathcal{E}$, we have $r_1 = r_2$. Then $A = B$ and the proof of 3) is complete.

Now we claim that:

4) G is an onto function.

To show 4) let $p \in X$. Then

$$p \in (E(X) - E_a(X)) \cup E_a(X) \cup R(X) \cup O(X).$$

Assume first $p \in E(X) - E_a(X)$. Let $r_p \in R(X)$ be such that $[p, r_p] \in \mathcal{E}$. Then $[p, r_p] \in \mathrm{Cl}_{C(X)}(\Omega(X)) - \Omega(X)$ and $G([p, r_p]) = p$. Assume now that $p \in R(X) \cup E_a(X)$. Then $\{p\} \in \mathrm{Cl}_{C(X)}(\Omega(X)) - \Omega(X)$ and $G(\{p\}) = p$. Assume now that $p \in O(X)$ and that, for every $[e,r] \in \mathcal{E}$, we have $p \notin [e,r]$. Then $\{p\} \in \Omega(X)$ and $G(\{p\}) = g(\{p\}) = p$. Finally assume that $p \in O(X)$ and that $p \in [e,r_e]$, for some $[e,r_e] \in \mathcal{E}$. If $p \in [s_e,r_e)$ then, by the definition of $f_1^{s_e}$, we have $t = (f_1^{s_e})^{-1}(p) \in [e,r_e)$. Thus, $A = \{t\} \in \Omega(X)$ and $G(A) = g(A) = f_1^{s_e}(t) = p$. If $p \in (e,s_e)$ and we let $q = (f_2^{s_e})^{-1}(p)$, then $q \in (e,r_e)$, $[e,q] \in \Omega(X)$ and $G([e,q]) = g([e,q]) = f_2^{s_e}(q) = p$. This shows that G is onto.

Now we claim that

5) G is a continuous function in $\Omega(X)$.

To show 5) let $A \in \Omega(X)$ and $\varepsilon > 0$. We need to find $\delta > 0$ such that

$$G\big(B_{C(X)}(A,\delta) \cap \mathrm{Cl}_{C(X)}\big(\Omega(X)\big)\big) \subset B_X\big(G(A),\varepsilon\big). \tag{4.1}$$

Assume first that $A = \{a\}$, for some $a \in O(X)$ and that, for every $[e,r] \in \mathcal{E}$, we have $a \notin [e,r]$. Then $G(A) = g(A) = a$. Since X is locally connected and $O(X)$ is open in X (Corollary 3.13), there is an open and connected subset U of X such that $a \in U \subset O(X)$. If U intersects some element $[e,r] \in \mathcal{E}$, then $r \in U$, and this contradicts the fact that $U \subset O(X)$. Hence, U does not intersect any element of $\mathcal{E}$. Let $\delta > 0$ be such that $\delta < \varepsilon$ and $B_X(a,\delta) \subset U$. If $B \in B_{C(X)}(A,\delta) \cap \mathrm{Cl}_{C(X)}(\Omega(X))$, then $B \subset N(\delta,A) = B_X(a,\delta) \subset U \subset O(X)$ so, by Theorem 4.8, $B = \{b\}$ for some $b \in O(X)$ and, for every $[e,r] \in \mathcal{E}$, we have $b \notin [e,r]$. Then $B \in \Omega(X)$, so $G(B) = g(B) = b$ and $G(B) \in B_X(a,\delta) \subset B_X(G(A),\varepsilon)$. Then (4.1) is satisfied.

Now assume that $A = \{q\}$, for some $q \in O(X)$ so that $q \in [e,r_e]$ and $[e,r_e] \in \mathcal{E}$. Then $G(A) = g(A) = f_1^{s_e}(q)$. Since $[e,r_e]$ is a free arc in X, there is $\delta_1 > 0$ such that $B_X(q,\delta_1) \subset (e,r_e)$. Since the function $f_1^{s_e}$ is continuous, there exists $\delta > 0$ such that $\delta < \delta_1$ and

$$f_1^{s_e}\big(B_X(q,\delta)\big) \subset B_X\big(f_1^{s_e}(q),\varepsilon\big).$$

If $B \in B_{C(X)}(A,\delta) \cap \mathrm{Cl}_{C(X)}(\Omega(X))$, then $B \subset N(\delta,A) = B_X(q,\delta) \subset B_X(q,\delta_1) \subset (e,r_e)$ so, by Theorem 4.8, $B = \{b\}$, for some $b \in B_X(q,\delta)$. Then $B \in \Omega(X)$, so $G(B) = g(B) = f_1^{s_e}(b) \in B_X(f_1^{s_e}(q),\varepsilon) = B_X(G(A),\varepsilon)$, and (4.1) is satisfied.

Now assume that $A = \{e\}$, for some $e \in E(X) - E_a(X)$. Let $r_e \in R(X)$ be so that $[e,r_e] \in \mathcal{E}$. Then, $G(A) = g(\{e\}) = f_1^{s_e}(e) = s_e$. Since $[e,r_e]$ is a free arc in X, there is $\delta_1 > 0$ such that $B_X(e,\delta_1) \subset [e,r_e)$. Since the functions $f_1^{s_e}$ and $f_2^{s_e}$ are continuous, there exists $\delta > 0$ such that $\delta < \delta_1$,

$$f_1^{s_e}\big(B_X(e,\delta)\big) \subset B_X\big(f_1^{s_e}(e),\varepsilon\big) = B_X(s_e,\varepsilon)$$

and

$$f_2^{s_e}\big(B_X(e,\delta)\big) \subset B_X\big(f_2^{s_e}(e),\varepsilon\big) = B_X(s_e,\varepsilon).$$

If $B \in B_{C(X)}(A,\delta) \cap \mathrm{Cl}_{C(X)}(\Omega(X))$, then $B \subset N(\delta,A) = B_X(e,\delta) \subset B_X(e,\delta_1) \subset [e,r_e)$. Thus, by Theorem 4.8, either $B = \{b\}$ for some $b \in B_X(e,\delta)$ or $B = [e,p]$, for some $p \in B_X(e,\delta)$. Then $B \in \Omega(X)$. Moreover, in the first case, $G(B) = g(\{b\}) = f_1^{s_e}(b) \in B_X(s_e,\varepsilon)$ and, in the second case, $G(B) = g([e,p]) = f_2^{s_e}(p) \in B_X(s_e,\varepsilon)$. Thus, $G(B) \in B_X(G(A),\varepsilon)$, so (4.1) is satisfied.

Now assume that there exists $[e,r_e] \in \mathcal{E}$ and $p \in (e,r_e)$ such that $A = [e,p] \subset [e,r_e]$. Then $G(A) = g(A) = f_2^{s_e}(p)$. Let $\delta_1 > 0$ be so that $N(\delta_1,A) \subset [e,r_e)$, $B_X(e,\delta_1) \cap B_X(p,\delta_1) = \emptyset$ and $B_X(p,\delta_1) \subset [e,r_e)$. Since the function $f_2^{s_e}$ is continuous, there exists $\delta > 0$ such that $\delta < \delta_1$ and

$$f_2^{s_e}\big(B_X(p,\delta)\big) \subset B_X\big(f_2^{s_e}(p),\varepsilon\big).$$

If $B \in B_{C(X)}(A,\delta) \cap \mathrm{Cl}_{C(X)}(\Omega(X))$, then $B \subset N(\delta,A) \subset N(\delta_1,A) \subset [e,r_e)$, $B_X(e,\delta) \cap B \neq \emptyset$ and $B_X(p,\delta) \cap B \neq \emptyset$. Hence, by Theorem 4.8, $B = [e,r]$ for some $r \in (e,r_e)$. Then $B \in \Omega(X)$, so $G(B) = g(B) = f_2^{s_e}(r)$ and, since $r \in B_X(p,\delta)$, we have $G(B) = f_2^{s_e}(r) \in B_X(f_2^{s_e}(p),\varepsilon) = B_X(G(A),\varepsilon)$, so (4.1) is satisfied. This completes the proof of 5).

Now we claim that:

6) G is a continuous function in $\mathcal{E}$.

To show 6) let $A \in \mathcal{E}$ and $\varepsilon > 0$. We need to find $\delta > 0$ such that (4.1) holds. Let $[e,r_e] \in \mathcal{E}$ be such that $A = [e,r_e]$. Then $G(A) = e$. Let $\delta_1 > 0$ be such that $B_X(e,\delta_1) \subset [e,r_e)$ and $B_X(e,\delta_1) \cap B_X(r_e,\delta_1) = \emptyset$. Since the function $f_2^{s_e}$ is continuous, there exists $\delta > 0$ such that $\delta < \delta_1$ and

$$f_2^{s_e}\big(B_X(r_e,\delta)\big) \subset B_X\big(f_2^{s_e}(r_e),\varepsilon\big) = B_X(e,\varepsilon).$$

If $B \in B_{C(X)}(A,\delta) \cap \mathrm{Cl}_{C(X)}(\Omega(X))$, then $e, r_e \in A \subset N(\delta,B) \subset N(\delta_1,B)$ so $B \cap B_X(e,\delta_1) \neq \emptyset$ and $B \cap B_X(r_e,\delta_1) \neq \emptyset$. This implies that B is a nondegenerate element of $\mathrm{Cl}_{C(X)}(\Omega(X))$ that intersects $[e,r_e]$. Thus, by Theorem 4.8, $B \subset [e,r_e]$ and there exists $p \in B_X(r_e,\delta)$ such that $B = [e,p] \subset [e,r_e]$. Then either $G(B) = e$ if $p = r_e$ or $G(B) = g(B) = f_2^{s_e}(p)$, if $p \neq r_e$. In both cases we have $G(B) \in B_X(G(A),\varepsilon)$, so (4.1) is satisfied. This completes the proof of 6).

Now we are going to prove that:

7) G is a continuous function in $F_1(R(X))$.

To show 7) let $A \in F_1(R(X))$ and assume that A is the limit, in the Hausdorff metric, of the sequence $\{A_n\}_{n\in\mathbb{N}}$ in $\mathrm{Cl}_{C(X)}(\Omega(X))$. Let us consider that, for infinitely many elements n, A_n satisfies either (b) or (d) of Theorem 4.8 or $A_n \in F_1(E(X))$. Then, for such indices n, we have $A_n \cap E(X) \neq \emptyset$ so, by Theorem 3.11, $A \cap E(X) \neq \emptyset$. Since this contradicts the fact that $A \in F_1(R(X))$ we can assume that, for every $n \in \mathbb{N}$, A_n satisfies either (a) or (c) and even more, that $A_n \in F_1(R(X) \cup O(X))$. Since by Theorem 3.12 every limit of ramification points of X is an end point of X, we can assume that $A_n \in F_1(O(X))$ for each $n \in \mathbb{N}$. Let $A = \{p\}$ and, for $n \in \mathbb{N}$, let $A_n = \{p_n\}$. Note that $G(A) = p$ and that p is the limit of the sequence $\{p_n\}_{n\in\mathbb{N}}$. If for infinitely many indices n, p_n is not an element of any member of $\mathcal{E}$ then, for such indices n, we have $G(A_n) = g(A_n) = p_n$. This implies that the sequence $\{G(A_n)\}_n$ of such indices n, converges to $G(A)$.

Now assume that p is so that $[e, p] \in \mathcal{E}$ for some $e \in E(X) - E_a(X)$. If for infinitely many indices n, we have $p_n \in [e, p]$ then, for such indices n, $G(A_n) = g(A_n) = f_1^{Se}(p_n)$. By the continuity of f_1^{Se}, the sequence $\{f_1^{Se}(p_n)\}_n$ of such indices n converges to $f_1^{Se}(p) = p = G(A)$. From this, and the fact that the order of p in X is finite, it follows that $G(A)$ is the limit of the sequence $\{G(A_n)\}_{n\in\mathbb{N}}$. This shows 7).

Now we claim that:

8) G is a continuous function in $F_1(E_a(X))$.

To show 8) let $A \in F_1(E_a(X))$ and assume that A is the limit, in the Hausdorff metric, of the sequence $\{A_n\}_{n\in\mathbb{N}}$ in $\mathrm{Cl}_{C(X)}(\Omega(X))$. Let $p \in E_a(X)$ be such that $A = \{p\}$. Then $G(A) = p$. By Theorem 3.7, $X = D_1 \cup D_2 \cup \cdots \cup D_l$, where each $D_i \in \mathfrak{D}$ and $|D_i \cap D_j| < \infty$ for every $i, j \in \{1, 2, \ldots, l\}$ with $i \neq j$. Since $p \in E(X) \subset X - P_X$, there exists $i \in \{1, 2, \ldots, l\}$ such that $p \in D_i - P_X$. By Theorem 3.8, $D_i - P_X$ is open in X, so $\mathcal{U} = \{B \in C(X)\colon\ B \subset D_i - P_X\}$ is an open subset of $C(X)$ that contains A. Hence, we can assume that $A_n \subset D_i - P_X$ for every $n \in \mathbb{N}$. Let $\{n_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{N}$. Let us assume that, for every $k \in \mathbb{N}$, we have $A_{n_k} = \{p_k\}$ and that either $p_k \in R(X) \cup E_a(X)$ or $p_k \in O(X)$ and p_k is not an element of any member of $\mathcal{E}$. Then p is the limit of the sequence $\{p_k\}_{k\in\mathbb{N}}$ and $G(A_{n_k}) = p_k$, for every $k \in \mathbb{N}$. This implies that $G(A)$ is the limit of the sequence $\{G(A_{n_k})\}_{k\in\mathbb{N}}$.

Now assume that A_{n_k} is contained in some member $[e_k, r_k]$ of $\mathcal{E}$, for every $k \in \mathbb{N}$. Then, $G(A_{n_k}) \in [e_k, r_k]$, for every $k \in \mathbb{N}$. Note that each $[e_k, r_k]$ is a connected subset of X that intersects D_i. Since P_X is finite and every point of X is of finite order in X, it follows that only for finitely many indices k, we have $[e_k, r_k] \not\subseteq D_i$. Then, we can consider that $[e_k, r_k] \subset D_i$, for each $k \in \mathbb{N}$. We can also consider that the sequence $\{e_k\}_{k\in\mathbb{N}}$ converges to $e \in D_i$. Given $k \in \mathbb{N}$ let us consider the arc $[e, e_k]$ in D_i. Since $e_k \in E(X)$ and $(e_k, r_k) \subset O(X)$, we have $r_k \in [e, e_k]$. Then, $A_{n_k} \cup \{G(A_{n_k})\} \subset [e_k, r_k] \subset [e, e_k]$. By property (S) described on p. 2072, the sequence of arcs $\{[e, e_k]\}_{k\in\mathbb{N}}$ converges, in the Hausdorff metric, to $\{\{e\}\}$. This implies that $e = p$ and that $G(A) = p$ is the limit of the sequence $\{G(A_{n_k})\}_{k\in\mathbb{N}}$. This shows 8).

To end the proof note that, by 3)–8), $G : \mathrm{Cl}_{C(X)}(\Omega(X)) \to X$ is a homeomorphism. Thus, $\mathrm{Cl}_{C(X)}(\Omega(X))$ is homeomorphic to X. $\square$

5. Main theorem

In this section we will show that if $X \in \mathfrak{LD}$ is different from an arc and a simple closed curve, then X has unique hyperspace $C(X)$.

Theorem 5.1. *Let X and Y be two continua different from an arc, and such that $C(X)$ is homeomorphic to $C(Y)$. If $X \in \mathfrak{LD}$, then X is homeomorphic to Y.*

Proof. By Theorem 2.2, Y is locally connected and, by Theorem 4.10, we have:

1) $\mathrm{Cl}_{C(X)}(\Omega(X))$ is homeomorphic to X.

Since $C(X)$ is homeomorphic to $C(Y)$, by Theorem 4.1, $\Omega(X)$ is homeomorphic to $\Omega(Y)$. Thus,

2) $\mathrm{Cl}_{C(X)}(\Omega(X))$ is homeomorphic to $\mathrm{Cl}_{C(Y)}(\Omega(Y))$.

Now we claim that:

3) Y contains at most a finite number of simple closed curves.

To show 3) assume first that S_1 and S_2 are two different simple closed curves in Y. By Theorem 4.5, we have $F_1(Y) \subset \mathrm{Cl}_{C(Y)}(\Omega(Y))$. Thus, $F_1(S_1) \cup F_1(S_2) \subset \mathrm{Cl}_{C(Y)}(\Omega(Y))$, so $F_1(S_1)$ and $F_1(S_2)$ are different subsets of $\mathrm{Cl}_{C(Y)}(\Omega(Y))$. By 1) and 2), $\mathrm{Cl}_{C(Y)}(\Omega(Y))$ is homeomorphic to X. Hence, there exist $A_1, A_2 \subset X$ such that $F_1(S_1)$ is homeomorphic to A_1 and $F_1(S_2)$ is homeomorphic to A_2. Since $F_1(S_1) \neq F_1(S_2)$, we have $A_1 \neq A_2$. Since S_i is homeomorphic to $F_1(S_i)$, for each $i = 1, 2$, we conclude that A_1 and A_2 are two different simple closed curves in X. We have shown that if Y contains two different simple closed curves, then X contains two different simple closed curves as well. Since X is a local dendrite, by Theorem 3.5, X contains at most a finite number of simple closed curves. Thus, Y contains at most a finite number of simple closed curves. This shows 3).

Since Y is a locally connected continuum with at most a finite number of simple closed curves, by Theorem 3.5, Y is a local dendrite. Hence, by Theorem 3.21, $Y \in \mathfrak{LD}$. Applying Theorem 4.10, it follows that $\mathrm{Cl}_{C(Y)}(\Omega(Y))$ is homeomorphic to Y. Using this, 1) and 2), we conclude that X is homeomorphic to Y. □

Corollary 5.2. *If $X \in \mathfrak{LD}$ is different from an arc and a simple closed curve, then X has unique hyperspace $C(X)$.*

Proof. Let X be as assumed, and Y be a continuum such that $C(X)$ is homeomorphic to $C(Y)$. If Y is either an arc or a simple closed curve then, by [2, Lemma 11, p. 38], X is either an arc or a simple closed curve. Since this is a contradiction, Y is different from an arc and a simple closed curve. Thus, by Theorem 5.1, X is homeomorphic to Y. □

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